Mathematical Logic

Lecture 10: Soundness and Completeness

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Equivalence of Formal Systems

Theorem

For any set of formulas Σ and formula φ,

$$\Sigma \vdash_H \varphi \iff \Sigma \vdash_{ND} \varphi$$

Equivalence of Formal Systems

- We have already established this equivalence for propositional logic.
- Therefore, all propositional logic axioms (P1, P2, P3) can be proved in ND, and all propositional ND rules are admissible in H.
- We will focus on the equivalence regarding the quantifier rules and axioms, as well as the equality symbol.

$H \subseteq ND$

Lemma

Every axiom of H is provable in ND, and every deduction rule of H is valid in ND.

Proof Sketch (Axiom 2): $\vdash_{ND} \forall x\varphi \to \varphi^x_t$

$$\dfrac{\dfrac{[\forall x\varphi]}{\varphi^x_t}\ \forall\ \text{Elim}}{\forall x\varphi \to \varphi^x_t}\ \to \text{Intro}$$

The term t is free for x in φ, so the substitution is valid.

$H \subseteq ND$

Proof Sketch (Axiom 3): $\vdash_{ND} \forall x(\varphi \to \psi) \to (\forall x\varphi \to \forall x\psi)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\forall x(\varphi \to \psi)]}{\varphi \to \psi}\ \forall\ \text{Elim} \qquad \dfrac{[\forall x\varphi]}{\varphi}\ \forall\ \text{Elim}}{\psi}\ \to \text{Elim}}{\forall x\psi}\ \forall\ \text{Intro}}{\forall x\varphi \to \forall x\psi}\ \to \text{Intro}}{\forall x(\varphi \to \psi) \to (\forall x\varphi \to \forall x\psi)}\ \to \text{Intro}$$

Notice that the application of $\forall$ Intro is valid because x does not occur free in the unclosed assumptions (which are at that point $\forall x(\varphi \to \psi)$ and $\forall x\varphi$, neither of which has free occurrences of x).

$H \subseteq ND$

Proof Sketch (Axiom 4): If x does not occur freely in φ, then $\vdash_{ND} \varphi \to \forall x\varphi$

$$\dfrac{\dfrac{[\varphi]}{\forall x\varphi}\ \forall\text{ Intro}}{\varphi \to \forall x\varphi}\ \to\text{ Intro}$$

The $\forall$ Intro step is valid precisely because of the restriction: x does not occur freely in φ (which is the only unclosed assumption at that point).

$H \subseteq ND$

Deduction Rules:

- Modus Ponens (MP): $\Sigma \vdash_{ND} \varphi$ and $\Sigma \vdash_{ND} \varphi \to \psi \implies \Sigma \vdash_{ND} \psi$. This is precisely the $\to$ Elimination rule in Natural Deduction.
- Universal Generalization (UG): If $\Sigma \vdash_{ND} \varphi$ and x does not occur freely in Σ, then $\Sigma \vdash_{ND} \forall x \varphi$. This is precisely the $\forall$ Introduction rule in Natural Deduction.

$H \subseteq ND$

Proof Sketch (Equality Axioms):

- $\vdash_{ND} x = x$ is a direct application of the $=$ Intro rule.
- $\vdash_{ND} x = y \to (\varphi \to \varphi')$ (where φ' replaces some free x's in φ with y)

$$\dfrac{\dfrac{\dfrac{[\varphi \equiv \psi^z_x] \qquad [x = y]}{\psi^z_y \equiv \varphi'} = \text{Elim}}{\varphi \to \varphi'} \to \text{Intro}}{x = y \to (\varphi \to \varphi')} \to \text{Intro}$$

Example (Remarks on $=$ Elim)

Suppose we have Pss and $s = t$, and we want to deduce Pst.

- We introduce a fresh variable z, and define $\psi(z)$ as Psz.
- Then ψ^z_s is exactly Pss.
- And ψ^z_t is exactly Pst.
- By applying $=$ Elim on ψ^z_s and $s = t$, we derive ψ^z_t, which is Pst.

$ND \subseteq H$

Lemma

Every Natural Deduction rule is admissible in the Hilbert-style axiomatic system.

$ND \subseteq H$

Theorem (Deduction Theorem for Predicate Logic)

For any set of formulas Γ and formulas φ, ψ:

$$\Gamma, \varphi \vdash_H \psi \iff \Gamma \vdash_H \varphi \to \psi$$

Proof sketch: The only non-trivial case is when $\Gamma, \varphi \vdash_H \forall x \psi$ is deduced by applying Universal Generalization

Theorem (Theorem on Constants)

If $\Gamma \vdash_H \varphi^x_c$, where c is a constant symbol that does not occur in Γ or in $\forall x \varphi$, then $\Gamma \vdash_H \forall x \varphi$.

Proof Sketch:

- By induction on the the proof sequence of φ^x_c from Γ.
- Since c does not occur in Γ, we uniformly replace all occurrences of c in the entire proof sequence with a completely fresh variable z, yielding $\Gamma \vdash_H \varphi^x_z$.

$ND \subseteq H$

Theorem (Theorem on Constants)

If $\Gamma \vdash_H \varphi^x_c$, where c is a constant symbol that does not occur in Γ or in $\forall x \varphi$, then $\Gamma \vdash_H \forall x \varphi$.

Proof Sketch:

- Since z is a variable and is not free in Γ, we apply the Universal Generalization (UG) rule to obtain $\Gamma \vdash_H \forall z \varphi^x_z$.
- Note that $(\varphi^x_z)^z_x = \varphi$. Therefore, $\Gamma \vdash \varphi$, which implies $\Gamma \vdash_H \forall x \varphi$.

$ND \subseteq H$

Simulating Quantifier Rules in H:

- $\forall$ **Elimination**: $\Gamma \vdash_H \forall x\varphi \implies \Gamma \vdash_H \varphi^x_t$.
  - We have Axiom 2: $\forall x\varphi \to \varphi^x_t$.
  - From $\Gamma \vdash_H \forall x\varphi$, apply Modus Ponens to obtain $\Gamma \vdash_H \varphi^x_t$.
- $\forall$ **Introduction**: $\Gamma \vdash_H \varphi^x_t \implies \Gamma \vdash_H \forall x\varphi$ (where t doesn't occur in Γ).
  - If t is a variable, this is an application of the Universal Generalization (UG) rule. If t is a constant, this follows directly from the **Theorem on Constants**.

$ND \subseteq H$

Simulating Existential Rules in H:

Recall that $\exists x \varphi$ can be viewed as an abbreviation for $\neg \forall x \neg \varphi$.

- $\exists$ **Introduction**: $\Gamma \vdash_H \varphi^x_t \implies \Gamma \vdash_H \exists x \varphi$.
  - This is equivalent to proving $\Gamma \vdash_H \varphi^x_t \to \neg \forall x \neg \varphi$.
  - This relies on Axiom 2 ($\forall x \neg \varphi \to \neg \varphi^x_t$) and propositional logic reasoning.

$ND \subseteq H$

- $\exists$ **Elimination**: If $\Gamma \vdash_H \exists x \varphi$ and $\Gamma, \varphi_t^x \vdash_H \psi$ (with t fresh).
  - By Deduction Theorem, $\Gamma \vdash_H \varphi_t^x \to \psi$, propositionally equivalent to $\Gamma \vdash_H \neg\psi \to \neg\varphi_t^x$.
  - By the **Theorem on Constants** (or UG if t is a variable), we deduce $\Gamma \vdash_H \forall x(\neg\psi \to \neg\varphi)$.
  - Using Axiom 3 and 4 (since x is not free in ψ), we derive $\Gamma \vdash_H \neg\forall x \neg\varphi \to \psi$.
  - Apply MP with $\exists x \varphi$ to obtain $\Gamma \vdash_H \psi$.

$ND \subseteq H$

Simulating Equality Rules in H:

- = **Introduction**: $\vdash_H t = t$.
  - The equality axiom specifies $x = x$. Substituting t for x yields $t = t$.
- = **Elimination**: $\Gamma \vdash_H \varphi^x_s$ and $\Gamma \vdash_H s = t \implies \Gamma \vdash_H \varphi^x_t$.
  - The equality axiom specifies $s = t \to (\varphi^x_s \to \varphi^x_t)$.
  - Applying Modus Ponens twice gives φ^x_t.

Soundness Theorem

Theorem (Soundness)

For any set of formulas Γ and formula φ, if $\Gamma \vdash \varphi$, then $\Gamma \vDash \varphi$.

- As we have shown $\Gamma \vdash_H \varphi \iff \Gamma \vdash_{ND} \varphi$, we can choose to prove soundness for either system.
- We sketch the proof for the Hilbert-style system by induction on the length of the proof sequence.

Soundness Theorem

- Base Case: We must show that all axioms are valid.
- Inductive Step: We must show that the deduction rules (Modus Ponens and Universal Generalization) preserve validity.

Soundness Theorem

Validity of Axiom 2: $\vDash \forall x \varphi \rightarrow \varphi^x_t$ (where t is free for x in φ)

Proof Sketch:

- Suppose $\mathcal{M} \vDash_s \forall x \varphi$.
- By definition, for *all* elements $a \in D$, $\mathcal{M} \vDash_{s[x:=a]} \varphi$.
- Let $a = [t]^{\mathcal{M}}_s$ be the evaluation of term t under the current assignment s.
- Then we have $\mathcal{M} \vDash_{s[x:=[t]^{\mathcal{M}}_s]} \varphi$.
- Since t is free for x in φ, Therefore, $\mathcal{M} \vDash_s \varphi^x_t$.

Soundness Theorem

Preservation of Validity (UG):

- Suppose $\Gamma \vdash \varphi$ and x does not occur freely in Γ.
- By the inductive hypothesis, $\Gamma \vDash \varphi$. We want to show $\Gamma \vDash \forall x \varphi$.
- Let $\mathcal{M}$ be a model and s be an assignment such that $\mathcal{M} \vDash_s \Gamma$. Since x is not free in Γ, for any $a \in D$, the modified assignment $s[x \mapsto a]$ agrees with s for any free variable in Γ, and so satisfies Γ.
- Therefore, by $\Gamma \vDash \varphi$, we have $\mathcal{M} \vDash_{s[x \mapsto a]} \varphi$ for all $a \in D$.

Completeness Theorem

Theorem (Completeness Theorem)

For any set of formulas Γ and formula φ, if $\Gamma \vDash \varphi$, then $\Gamma \vdash \varphi$.

- The Completeness Theorem is one of the most fundamental results in logic (proved by Kurt Gödel).
- It states that our formal deduction system is powerful enough to derive *all* semantic truths.
- To prove it, we usually rely on an equivalent formulation regarding consistent sets of formulas.

Consistency and Satisfiability

Definition (Consistency)

A set of formulas Γ is consistent if there is no formula ψ such that both $\Gamma \vdash \psi$ and $\Gamma \vdash \neg\psi$.

Theorem (Model Existence)

Every consistent set of formulas Γ is satisfiable (i.e., there exists a model $\mathcal{M}$ and assignment s such that $\mathcal{M} \vDash_s \Gamma$).

We will first prove that the Completeness Theorem and the Model Existence Theorem are logically equivalent.

Consistency and Satisfiability

Completeness $\implies$ Model Existence: Let Γ be a consistent set of formulas. We want to show Γ is satisfiable.

Suppose for contradiction that Γ is *not* satisfiable. Then there is no model $\mathcal{M}$ and assignment s such that $\mathcal{M} \vDash_s \Gamma$. Hence, we have $\Gamma \vDash \psi$ and $\Gamma \vDash \neg\psi$ for any formula ψ.

By the Completeness Theorem, this implies $\Gamma \vdash \psi$ and $\Gamma \vdash \neg\psi$.

But this contradicts the assumption that Γ is consistent.

Consistency and Satisfiability

Model Existence $\implies$ Completeness: Suppose $\Gamma \vDash \varphi$. We want to show $\Gamma \vdash \varphi$.

Suppose for contradiction that $\Gamma \nvdash \varphi$. If $\Gamma \nvdash \varphi$, then the set $\Gamma \cup \{\neg\varphi\}$ must be consistent.

By the Model Existence Theorem, $\Gamma \cup \{\neg\varphi\}$ is satisfiable. So there is a model $\mathcal{M}$ and assignment s such that $\mathcal{M} \vDash_s \Gamma$ and $\mathcal{M} \vDash_s \neg\varphi$. This contradicts our premise that $\Gamma \vDash \varphi$.

Proof of Model Existence

Goal: Prove that every consistent set Γ has a model $\mathcal{M}$.

Intuition: We can build a model out of the syntax of the language itself! This is called a Term Model (or Henkin Model).

- Domain: The set of all constant symbols C in the language. (We will discuss the case with equality $=$ shortly).

Proof of Model Existence

Goal: Prove that every consistent set Γ has a model $\mathcal{M}$.

Intuition: We can build a model out of the syntax of the language itself! This is called a Term Model (or Henkin Model).

- Interpretation: Constants denote themselves: $c^{\mathcal{M}} = c$.

  For relations, we let syntax directly define the semantics:

$$R^{\mathcal{M}}(c_1, \ldots, c_n) \text{ holds in } \mathcal{M} \iff R(c_1, \ldots, c_n) \in \Gamma$$

The Truth Lemma

If we define the model this way, we naturally want the following Truth Lemma to hold for any sentence φ:

$$\mathcal{M} \vDash \varphi \iff \varphi \in \Gamma$$

Let's test this by structural induction on φ.

- Atomic formulas: Holds directly by our definition

The Truth Lemma

- Negation ($\neg$): We want $\mathcal{M} \models \neg\psi \iff \neg\psi \in \Gamma$. By semantics, $\mathcal{M} \models \neg\psi \iff \mathcal{M} \not\models \psi \iff \psi \notin \Gamma$. This requires $\psi \notin \Gamma \iff \neg\psi \in \Gamma$.

Requirement 1: Γ must have a complete "opinion" on every formula. It needs to be a Maximal Consistent Set .

The Need for Witnesses

Let's test the Truth Lemma for the existential quantifier ($\exists$).

- Existential ($\exists$): We want $\mathcal{M} \models \exists x\psi(x) \iff \exists x\psi(x) \in \Gamma$.
- If $\mathcal{M} \models \exists x\psi(x)$, we hope there is some constant c such that $\mathcal{M} \models \psi(c)$. By the inductive hypothesis, $\psi(c) \in \Gamma$.
- This requires: if $\exists x\psi(x) \in \Gamma$, there must be some constant c such that $\psi(c) \in \Gamma$.

The Need for Witnesses

But a consistent set (like $\Gamma = \{\exists x Px\}$) might not mention any constants!

Requirement 2: We must expand our language with new constants and add Henkin axioms $\exists x\psi \to \psi(c)$ to provide a "witness" (name) for every existential formula.

Formal Proof Steps

To make the Truth Lemma actually work, we carry out these formal steps:

1. **Adding Witnesses (Henkin Theory).** Expand the language with a countably infinite set of new constants C. For every formula $\exists x \varphi(x)$, add an axiom $\exists x \varphi(x) \to \varphi(c)$ for a unique fresh constant $c \in C$. Let this new set be Γ_H.
   - *Fact:* If Γ is consistent, then Γ_H is still consistent.

Formal Proof Steps

To make the Truth Lemma actually work, we carry out these formal steps:

1. **Maximal Consistent Set.**
   - *Lindenbaum's Lemma:* Any consistent set Γ_H can be extended to a Maximal Consistent Set Γ^* (where for every sentence ψ, exactly one of $\psi \in \Gamma^*$ or $\neg\psi \in \Gamma^*$ holds).

Formal Proof Steps

3. **Construct the Term Model.** Build the term model $\mathcal{M}$ from the constants in the expanded language using the maximal set Γ^*.
4. **Truth Lemma.** Prove by induction that for any sentence φ, $\mathcal{M} \vDash \varphi \iff \varphi \in \Gamma^*$. (The maximality of Γ^* and the Henkin axioms ensure all inductive steps hold flawlessly!)
5. **Conclusion.** Since $\Gamma \subseteq \Gamma^*$, we have $\mathcal{M} \vDash \Gamma^*$, and thus $\mathcal{M} \vDash \Gamma$.

Therefore, Γ is satisfiable.

Handling Equality

The Problem with Equality: If our language includes the equality symbol =, we cannot simply let the domain be the set of constants C.

- If $(c = d) \in \Gamma^*$, our model *must* interpret c and d as the exact same object, i.e., $c^{\mathcal{M}} = d^{\mathcal{M}}$.
- But if they are distinct constant symbols, they are distinct syntactic objects in C $(c \neq d)$.

Handling Equality

The Solution: We define an equivalence relation $\sim$ on the set of constants C:

$$c \sim d \iff (c = d) \in \Gamma^*$$

The equality axioms in Γ^* (like $x = x$, $x = y \to y = x$, $x = y \to y = z \to x = z$) ensure that $\sim$ is reflexive, symmetric, and transitive.

Handling Equality

We then construct the Quotient Term Model :

- Domain: The set of equivalence classes $D = \{[c] \mid c \in C\}$.
- Interpretation of Constants: $c^{\mathcal{M}} = [c]$.
- Interpretation of Relations: $R^{\mathcal{M}}([c_1], \ldots, [c_n])$ holds $\iff R(c_1, \ldots, c_n) \in \Gamma^*$.

Handling Equality

Well-definedness: What if we picked different representatives $d_i \in [c_i]$? The equality substitution axiom $x_i = y_i \to (R \ldots x_i \ldots \to R \ldots y_i \ldots)$ guarantees that if $c_i \sim d_i$, then

$$R \ldots c_i \ldots \in \Gamma^* \iff R \ldots d_i \ldots \in \Gamma^*$$

So the interpretation of R does not depend on the choice of representatives!

Handling Equality

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$$R \ldots c_i \ldots \in \Gamma^* \iff R \ldots d_i \ldots \in \Gamma^*$$

So the interpretation of R does not depend on the choice of representatives!

Exercises & Discussion* 10.1

Let φ and φ' be alphabetic variants of each other (i.e., they only differ in the names of their bound variables).

Prove that:

$$\vdash \varphi \leftrightarrow \varphi'$$

Hint: You may first prove a simpler base case: $\vdash \forall x \psi \leftrightarrow \forall y \psi^x_y$.

Exercises & Discussion 10.2

In our treatment of equality, we mentioned that the equality axioms guarantee that equality behaves like an equivalence relation.

Using the equality axioms (or the = Intro and = Elim rules in Natural Deduction), prove the following properties for any terms t_1, t_2, t_3:

1. **Reflexivity:** $\vdash t_1 = t_1$
2. **Symmetry:** $\vdash t_1 = t_2 \rightarrow t_2 = t_1$
3. **Transitivity:** $\vdash t_1 = t_2 \rightarrow (t_2 = t_3 \rightarrow t_1 = t_3)$

Exercises & Discussion 10.3

Consider the following well-known theorem of classical predicate logic, sometimes playfully referred to as the Drinker Paradox :

> *"In any non-empty pub, there is a person such that if they are drinking, then everyone in the pub is drinking."*

Prove this statement in Natural Deduction (or the Hilbert system):

$$\vdash \exists x(Px \rightarrow \forall y Py)$$

Exercises & Discussion 10.4

Let Γ be a consistent set of formulas in first-order logic, and let φ be any sentence.

Prove that at least one of the following sets must be consistent:

$$\Gamma \cup \{\varphi\} \quad \text{or} \quad \Gamma \cup \{\neg\varphi\}$$