Mathematical Logic

Lecture 8: Syntax of Predicate Logic

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Syntax of Predicate Logic

Similar to propositional logic, we need to formally define what a well-formed formula of predicate logic is. This is called the syntax of a logic, analogous to the grammar of natural language.

Syntax of Predicate Logic

Basic Symbols

- Variables : $v_1, v_2, \ldots$
- Constants : $c_1, c_2, \ldots$
- Predicates : $P_1, P_2, \ldots$

  And we must specify the arity of each predicate.
- Quantifiers : $\exists$, $\forall$
- Propositional connectives, parentheses

Syntax of Predicate Logic

Convention

We often use the following meta-language symbols (and their subscripted versions) to refer to the basic symbols in the language of predicate logic:

- Use x, y, z for variables
- Use a, b, c for constants
- Use P, Q, R for predicates

Furthermore, both variables and constants are terms . We usually use t to refer to terms (variables or constants).

Syntax of Predicate Logic

Definition (Well-Formed Formula)

- If P is an n-ary predicate symbol and $t_1, \ldots, t_n$ are terms, then $Pt_1 \ldots t_n$ is an atomic formula . All atomic formulas are well-formed formulas (or formulas for short).
- If φ, ψ are formulas, then $(\neg\varphi)$, $(\varphi \to \psi)$, $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \leftrightarrow \psi)$ are formulas .
- If φ is a formula and x is a variable, then $\exists x\varphi$ and $\forall x\varphi$ are formulas .
- Nothing else is a formula.

Syntax of Predicate Logic

Example

Suppose P, Q are unary predicates, R is a binary predicate, and S is a ternary predicate. The following strings are well-formed formulas:

- The rules for omitting parentheses still apply
- $(Px \wedge Qc)$
- $(Px \wedge Qy) \vee Rxz$
- $\forall x Rxx$
- $\exists x(Rxy \wedge Sxyx)$

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Syntax of Predicate Logic

- What is the difference in syntactic function between variables and constants?
- Consider the formula: $Px \wedge \forall x(Qx \rightarrow Rxy)$
  We say that the variable x has different occurrences here.
  Intuitively, the different occurrences of x in the above formula have different semantics, thus have different syntactic roles.
- Similarly, we can refer to different occurrences of any string in a given formula.

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Syntax of Predicate Logic

Definition (Bound and Free Occurrences of Variables)

- We call this φ in the formula $\forall x\varphi$ the **scope** of this $\forall x$. Every free occurrence of x in the scope of a certain $\forall x$ is a **bound occurrence** (bound by this $\forall x$).
- $\exists x\varphi$ is similar.
- If an occurrence of a variable x is neither a bound occurrence nor the occurrence within some $\forall x$ or $\exists x$, then it is a **free occurrence**.

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Syntax of Predicate Logic

Example

- In Rxy, both x and y are free occurrences.
- In $\exists xRxy$, the scope of the $\exists x$ is Rxy; x in $\exists xRxy$ is a bound occurrence (bound by that $\exists x$); the occurrence of x in the subformula Rxy is a free occurrence.
- $Px \to \exists xRxy$
- $\forall x\forall y(Px \to \exists xRxy)$

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Syntax of Predicate Logic

Notation

- If a formula contains at least one free occurrence of a variable, we call this formula an **open formula**.
- Otherwise, we call it a **closed formula**. A closed formula is usually also called a **sentence**.

Syntax of Predicate Logic

Example

- $\forall x(Px \to Rxy)$
- $\exists x(Px \wedge \exists xRxx)$
- $\forall xRxc$

Syntax of Predicate Logic

Sometimes the scope of a certain $\forall x$ does not contain any free x. In this case, we call this $\forall x$ a **vacuous** quantifier (occurrence).

Example

- $\forall x \exists y Rxx$
- $\forall x \exists x Rxx$
- $\forall x \exists y Ryy$
- $\forall x (Px \rightarrow \exists x Qx)$

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Syntax of Predicate Logic

Definition (Substitution)

We define the substitution operation on strings. First consider terms:

- $c^c_t = t$
- If $t_1 \neq c$, then $c^{t_1}_{t_2} = c$
- $x^x_t = t$
- If $t_1 \neq x$, then $x^{t_1}_{t_2} = x$

Syntax of Predicate Logic

Next, we define substitution for formulas. Intuitively, we want φ_t^x to represent replacing all free occurrences of x in φ with the term t. The strict recursive definition is given below:

Syntax of Predicate Logic

Definition (Substitution)

- For atomic formulas, $(Pt_1 \ldots t_n)_t^x = P(t_1)_t^x \ldots (t_n)_t^x$
- $(\neg\varphi)_t^x = (\neg\varphi_t^x)$
- $(\varphi * \psi)_t^x = (\varphi_t^x * \psi_t^x)$ (where $*$ can be $\rightarrow, \wedge, \vee, \leftrightarrow$)
- $(Qy\varphi)_t^x = \begin{cases} Qy\varphi, & \text{if } x = y, \\ Qy\varphi_t^x, & \text{if } x \neq y \end{cases}$ (where Q can be $\forall$ or $\exists$)

Syntax of Predicate Logic

Example

- $(Rxx)^x_c = Rcc$
- $(Rxc)^c_x = Rxx$
- $(\forall y Rxy)^x_z = \forall y Rzy$

Syntax of Predicate Logic

- Intuitively, $\forall x P x$ and $\forall y P y$ have the same meaning.
- Consider $(\forall y R x y)^x_y = \forall y R y y$, which has a significantly different meaning from $(\forall y R x y)^x_z = \forall y R z y$. The reason for this difference seems to be an unfortunate choice of variables.
- $\forall y R x y$ and $\forall z R x z$ mean the same thing, but $(\forall z R x z)^x_y$ has no such issue.

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- Consider $(\forall y Rxy)^x_y = \forall y Ryy$, which has a significantly different meaning from $(\forall y Rxy)^x_z = \forall y Rzy$. The reason for this difference seems to be an unfortunate choice of variables.
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- Intuitively, $\forall x P x$ and $\forall y P y$ have the same meaning.
- Consider $(\forall y R x y)_y^x = \forall y R y y$, which has a significantly different meaning from $(\forall y R x y)_z^x = \forall y R z y$. The reason for this difference seems to be an unfortunate choice of variables.
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Definition (Alphabetic Variant)

Suppose z does not occur in the formula φ. We call $\forall z \varphi^x_z$ a (bound) alphabetic variant of $\forall x \varphi$. The case for $\exists$ is similar.

If φ' is obtained from φ by replacing subformulas with their alphabetic variants finitely many times, we also call φ' an alphabetic variant of φ.

Syntax of Predicate Logic

Example

- $\forall z Rxz$ is an alphabetic variant of $\forall y Rxy$.
- $\forall x(Px \rightarrow \exists x Rxx)$ has alphabetic variants such as $\forall y(Py \rightarrow \exists x Rxx)$ or $\forall x(Px \rightarrow \exists y Ryy)$.
- Is $\forall y(Py \rightarrow \exists y Ryy)$ an alphabetic variant of $\forall x(Px \rightarrow \exists y Rxy)$?

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Exercises & Discussion

- Which occurrences of x in the following formula are bound occurrences?

  $\exists x(Rxy \lor Sxyz) \land Px$

- Which occurrences of quantifiers bind which occurrences of variables below?

  $\forall x(Px \to \exists xRxx)$

Exercises & Discussion

Which of the following formulas are sentences?

- $\forall x Px \lor \forall x Qx$
- $Px \lor \forall x Qx$
- $\forall x(Px \rightarrow Rcx)$
- $\forall x(Px \rightarrow \exists y Rxy)$

Exercises & Discussion

- Is the "alphabetic variant" relation symmetric? If φ is an alphabetic variant of ψ, is ψ necessarily an alphabetic variant of φ?
- Is the "alphabetic variant" relation transitive?

Exercises & Discussion *

We use quantifier depth to indicate the number of nested levels of quantifiers. For example, the atomic formula Px has a quantifier depth of 0, $\forall x Rxy$ has a quantifier depth of 1, and $\exists y(Py \wedge \forall x Rxy)$ has a quantifier depth of 2. Try to give a recursive definition of the quantifier depth for predicate logic formulas.