Mathematical Logic

Lecture 7: Predicate Logic

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Predicate Logic

- Propositional logic ignores the internal structure of basic propositions.

Example

- "Adam is walking" is represented as p.
- "Adam is standing upside down" is represented as q.

- The information conveyed by basic propositions is lost.

Predicate Logic

- In predicate logic, we use variables / constants and predicates to represent objects and properties respectively.

Example

- "Adam is walking" is represented as Wa or $W(a)$.

  i.e., the object Adam has the property "walking".
- "Adam is standing upside down" is represented as Da or $D(a)$.

- What is the advantage over propositional logic?

Predicate Logic

- Suppose we have Adam (a), Bob (b), Charles (c)...
- We use propositional variables $p_1, p_2, p_3, p_4, p_5, p_6 \ldots$ to represent $Wa, Da, Wb, Db, Wc, Dc, \ldots$ respectively.
- Suppose we know that a person cannot be walking and standing upside down at the same time, we naturally get $\neg(Wa \wedge Da)$, $\neg(Wb \wedge Db)$, $\neg(Wc \wedge Dc), \ldots$
- Another powerful component of predicate logic, quantifiers , allows us to express "a person cannot be walking and standing upside down at the same time."

Predicate Logic

The Language of Predicate Logic

- Constants : $a, b, c, \ldots$
- Variables : $x, y, z, \ldots$
- Predicates : $P, Q, R, \ldots$

  Predicates are classified as unary, binary, ternary...

The Language of Predicate Logic

Example

- Adam is walking: Wa
- Adam sees Bob: Sab (binary predicate)
- Adam introduces Bob to Charles: $Iabc$ (ternary predicate)

Predicate Logic

The Language of Predicate Logic

- Propositional connectives: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$
- Quantification : $\forall, \exists$. Each quantifier always carries a variable.

Example

- $\exists x W x$
- $\forall x (W x \rightarrow \neg D x)$

Predicate Logic

- To make it easier to remember, we usually use uppercase letters for predicates, and try to match natural language (English or Chinese). For example, use W for "walk", and D for " 倒立" (stand upside down). The specific choice doesn't matter, just declare it before translating.
- In mathematics, we might encounter many more predicates, and we can even use $P_1, P_2, \ldots$ to represent infinitely many predicates.

Predicate Logic

- The order of constants or variables after a binary or n-ary predicate is important. Sab (Adam sees Bob) and Sba (Bob sees Adam) are obviously different.
- We rarely encounter predicates with an arity greater than 3 in daily language, but in mathematics, we can have arbitrary n-ary predicates.

Predicate Logic

- Sometimes some predicates can be defined by other predicates.

Example

- Let Dxy mean "x and y are of opposite sex", Fx mean "x is female", and Mx mean "x is male". Then Dxy if and only if $(Fx \rightarrow My) \wedge (Mx \rightarrow Fy)$.

Predicate Logic

Example

- Let $Mxyz$ mean "x is between y and z", and $x < y$ mean "x is less than y". Then $Mxyz$ if and only if $(y < x \wedge x < z) \vee (z < x \wedge x < y)$.

- Note that sometimes, to conform to natural language reading habits, for some binary predicates, we write formulae like $< xy$ as $x < y$.

Predicate Logic

- Constants correspond to proper names or special nouns in daily language, like "Chiang Kai-shek", "Jerusalem", or "0", "π" in mathematics...
- Variables have no strict equivalent in daily language, but they act similarly to pronouns. For example: "Adam saw that person, and he was standing upside down": $Sax \wedge Dx$.

Predicate Logic

Example

- $2 < 5$, where 2 and 5 are constants. The meaning of the proposition is clear, and its truth value is definite.
- $x < y$, its meaning and truth value depend on the values of x and y.

  In this sense, $x < y$ or $x < 5$ is not a complete sentence.

Predicate Logic

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Predicate Logic

Example

- Sax, "Adam saw it/him"
- Sxy, "It/He saw it/him"
- Sxx, "It/He saw itself/himself"

Predicate Logic

Composition of Propositions in Predicate Logic

Example

- Adam did not see Bob: $\neg Sab$
- Adam saw Bob or Charles: $Sab \vee Sac$
- If Adam sees Bob, he will be happy: $Sab \rightarrow Ha$

Predicate Logic

Example (Usage of some quantifiers)

- Someone is walking $\exists x Wx$
- A boy is walking $\exists x(Bx \land Wx)$
- Adam sees a boy $\exists x(Bx \land Sax)$
- A boy sees Adam $\exists x(Bx \land Sxa)$
- A boy sees himself $\exists x(Bx \land Sxx)$

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Predicate Logic

Example (Usage of some quantifiers)

- Everyone is standing upside down $\forall x D x$
- All boys are standing upside down $\forall x(Bx \rightarrow Dx)$
- All boys saw Adam $\forall x(Bx \rightarrow Sxa)$

Predicate Logic

Example (Usage of some quantifiers)

- Everyone is standing upside down $\forall x Dx$
- All boys are standing upside down $\forall x(Bx \to Dx)$
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Predicate Logic

- You might have noticed that the existential quantifier $\exists x$ is often followed by a conjunction, while the universal quantifier $\forall x$ is often followed by an implication.
- Try to understand what the following two formulas mean:
  - $\forall x(Bx \land Dx)$
  - $\exists x(Bx \to Dx)$

Predicate Logic

Example (Nested quantifiers)

- Everyone has someone they love $\forall x \exists y Lxy$
- Someone loves everyone $\exists x \forall y Lxy$
- Everyone is loved by someone $\forall x \exists y Lyx$
- Someone is loved by everyone $\exists x \forall y Lyx$

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Predicate Logic

- Note that although there are many variables in the above examples, the meaning of the whole formula is definite (it does not depend on what x and y refer to).
- Try to interpret L as $>$ in natural numbers, understand $\forall x$ as "all natural numbers", and $\exists x$ as "there exists a natural number".What if we interpret it as "all/exists real numbers"?
- For a more precise translation, we always need to stipulate the domain of discourse of the quantifiers.

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Predicate Logic and Natural Language

Translating Syllogisms

- All A are B: $\forall x(Ax \to Bx)$
- No A are B: $\forall x(Ax \to \neg Bx)$
- Some A are B: $\exists x(Ax \land Bx)$
- Some A are not B: $\exists x(Ax \land \neg Bx)$

Predicate Logic and Natural Language

More predicates

Every boy loves a girl

- Every boy has the property "loves a girl": $\forall x(Bx \to \varphi(x))$
- $\varphi(x)$ represents "x loves a girl": $\exists y(Gy \wedge Lxy)$
- $\forall x(Bx \to \exists y(Gy \wedge Lxy))$

Predicate Logic and Natural Language

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Predicate Logic and Natural Language

Complex predicate logic formulas can also be analyzed by their construction trees.

$$\forall x(Bx \rightarrow \exists y(Gy \wedge Lxy))$$

$$Bx \rightarrow \exists y(Gy \wedge Lxy)$$

$$Bx \qquad \exists y(Gy \wedge Lxy)$$

$$Gy \wedge Lxy$$

$$Gy \qquad Lxy$$

Predicate Logic and Natural Language

Example

No girl who loves a boy is not loved by some boy

Predicate Logic and Natural Language

More nested quantifiers

Example

You can fool some people all the time,

and you can fool all the people some time,

but you cannot fool all the people all the time.

Predicate Logic and Natural Language

More nested quantifiers

Example (The function f is continuous at x)

For any positive real number ε, there exists a real number δ, such that for any real number y satisfying $|x - y| < \delta$, we have $|f(x) - f(y)| < \varepsilon$.

Next, we will observe some reasoning in predicate logic based on intuition.

Monadic Predicate Logic

Monadic Predicate Logic is a fragment of predicate logic that contains only unary predicates (it does not contain constants, function symbols, binary or higher-arity predicate symbols, equality, etc.)

- Syllogisms can be translated into the language of monadic predicate logic.
- We can use Venn diagrams to help determine whether a monadic predicate logic inference is valid.

Monadic Predicate Logic

Consider the following statements:

- Some A are not B
- Not all A are B
- No A are B
- No A are B

Monadic Predicate Logic

Consider the following statements:

- $\exists x(Ax \wedge \neg Bx)$
- Not all A are B
- No A are B
- No A are B

Monadic Predicate Logic

Consider the following statements:

- $\exists x(Ax \wedge \neg Bx)$
- $\neg\forall x(Ax \rightarrow Bx)$
- No A are B
- No A are B

Monadic Predicate Logic

Consider the following statements:

- $\exists x(Ax \wedge \neg Bx)$
- $\neg \forall x(Ax \rightarrow Bx)$
- $\forall x(Ax \rightarrow \neg Bx)$
- No A are B

Monadic Predicate Logic

Consider the following statements:

- $\exists x(Ax \land \neg Bx)$
- $\neg\forall x(Ax \to Bx)$
- $\forall x(Ax \to \neg Bx)$
- $\neg\exists x(Ax \land Bx)$

Monadic Predicate Logic

The above observations suggest the following equivalences:

- $\neg\forall x\varphi$ is equivalent to $\exists x\neg\varphi$
- $\neg\exists x\varphi$ is equivalent to $\forall x\neg\varphi$

Therefore, we also have:

- $\forall x\varphi$ is equivalent to $\neg\exists x\neg\varphi$
- $\exists x\varphi$ is equivalent to $\neg\forall x\neg\varphi$

Monadic Predicate Logic

Monadic predicate logic is "more" than syllogisms:

- $\forall x(Ax \wedge Bx)$ is equivalent to $\forall x Ax \wedge \forall x Bx$

  $\forall$ distributes over $\wedge$

- Is $\forall x(Gx \vee Bx)$ equivalent to $\forall x Gx \vee \forall x Bx$?

  $\forall$ does not distribute over $\vee$

What about $\exists$ with $\wedge$ and $\vee$?

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Reasoning Involving Polyadic Predicates

Example

All donkeys are animals, so all heads of donkeys are heads of animals.

$$\frac{\forall x(Dx \to Ax)}{\forall y(Hy \land \exists x(Dx \land Pxy) \to \exists z(Az \land Pzy \land Hy))}$$

Reasoning Involving Polyadic Predicates

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All donkeys are animals, so all heads of donkeys are heads of animals.

$$\frac{\forall x(Dx \rightarrow Ax)}{\forall y(Hy \wedge \exists x(Dx \wedge Pxy) \rightarrow \exists z(Az \wedge Pzy \wedge Hy))}$$

Reasoning Involving Polyadic Predicates

The following formulas are valid:

- $\forall x \forall y \varphi \leftrightarrow \forall y \forall x \varphi$
- $\exists x \forall y \varphi \rightarrow \forall y \exists x \varphi$

What about other cases of nested quantifiers?

Reasoning Involving Polyadic Predicates

The following formulas are valid:

- $\forall x \forall y \varphi \leftrightarrow \forall y \forall x \varphi$
- $\exists x \forall y \varphi \rightarrow \forall y \exists x \varphi$

What about other cases of nested quantifiers?

"Cases" or "Pictures" for Predicate Logic

- Like propositions, the concept of validity in predicate logic depends on our understanding of the situations involved.
- Predicate logic is more expressive, and the situations it involves can be very complex.
- However, the expressive power of predicate logic is limited in the sense that there can be many different situations that satisfy the same formulae.
- We need more abstract account of "situations".

"Cases" or "Pictures" for Predicate Logic

- Cases in monadic predicate logic can be abstracted as Venn diagrams.
- If only n predicates appear in a set of monadic predicate logic formulas, the corresponding case can be represented by a Venn diagram with 2^n regions.
- The validity of monadic predicate logic formulas is decidable.

"Cases" or "Pictures" for Predicate Logic

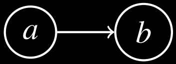
- People often use pictures to illustrate a specific situation.
- A situation corresponding to a set of predicate logic formulas containing only one binary predicate symbol can be represented by a graph.

Definition (Graph)

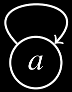
A graph or directed graph consists of a set of vertices (points, nodes) and arrows (edges, arcs) between the vertices.

"Cases" or "Pictures" for Predicate Logic

Graph:



Rab



Raa

"Cases" or "Pictures" for Predicate Logic

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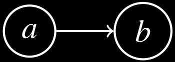

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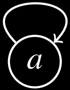

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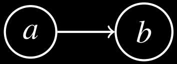

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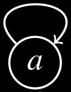

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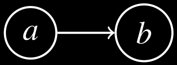
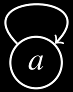


$\exists x \exists y Rxy$

$\neg \forall x \exists y Rxy$

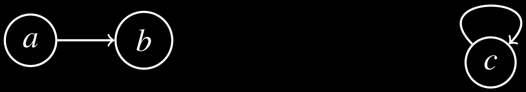


$\exists x Rxx$

$\forall x \exists y Rxy$

"Cases" or "Pictures" for Predicate Logic

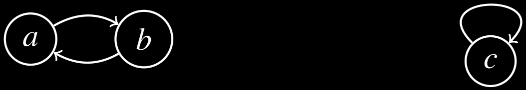
Consider the following as a single entire graph:



$$\exists x \exists y Rxy,\ \exists x Rxx,\ \forall x \exists y Rxy$$

"Cases" or "Pictures" for Predicate Logic

Add an arrow:

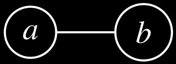


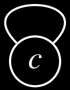
$\forall x(Rxy \to Ryx)$ Symmetry

We call a graph that satisfies symmetry an undirected graph .

"Cases" or "Pictures" for Predicate Logic

We can simplify the drawing of undirected graphs:





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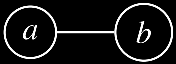
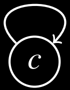
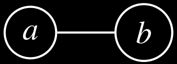




"Cases" or "Pictures" for Predicate Logic

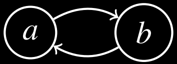
- We say a graph satisfying $\forall x Rxx$ is reflexive .
- A graph satisfying $\forall x \forall y \forall z (Rxy \rightarrow Ryz \rightarrow Rxz)$ is transitive .
- Are the following relations reflexive, transitive, or symmetric? In the domain of humans: "... is ...'s parents", "... is ...'s ancestors", "... is ...'s siblings".

"Cases" or "Pictures" for Predicate Logic



Is it transitive?

"Cases" or "Pictures" for Predicate Logic



Is it transitive?

"Cases" or "Pictures" for Predicate Logic

We can add more symbols beside a binary predicate:

- Add an equality symbol (equals sign)

  $\forall x \forall y (Rxy \vee Ryx \vee x = y)$ means R is linear .

- Add a unary predicate P

  $\forall x (Px \to \exists y (\neg Py \wedge Rxy))$

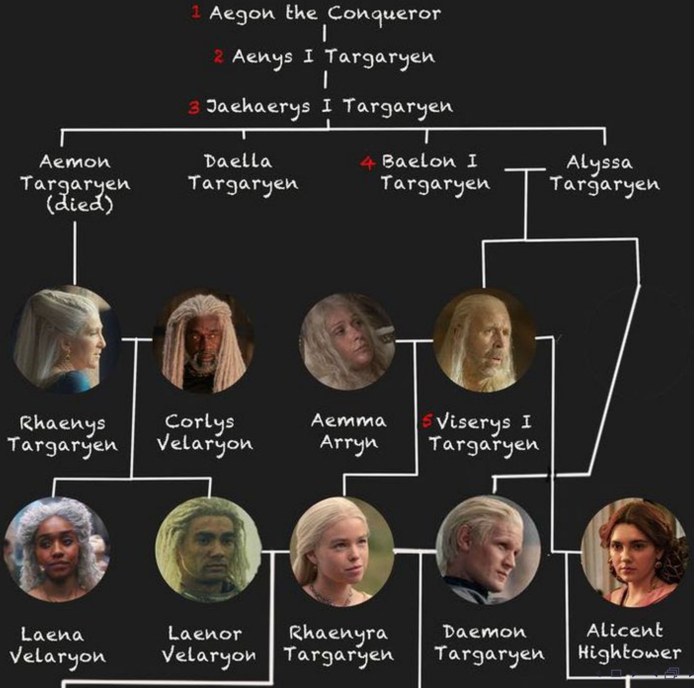
"Cases" or "Pictures" for Predicate Logic

Example (Family tree)

Let Pxy mean x is a parent of y, and Fx mean x is female.

Try to represent the following using predicate logic formulas:

- x is the father of y
- Aemma and Viserys are brother and sisters
- Daemon is an uncle of Rhaenyra


1 Aegon the Conqueror
2 Aenys I Targaryen
3 Jaehaerys I Targaryen
Aemon Targaryen (died)
Daella Targaryen
4 Baelon I Targaryen
Alyssa Targaryen
Rhaenys Targaryen
Corlys Velaryon
Aemma Arryn
5 Viserys I Targaryen
Laena Velaryon
Laenor Velaryon
Rhaenyra Targaryen
Daemon Targaryen
Alicent Hightower

"Cases" or "Pictures" for Predicate Logic

Example (Divisibility and primes in natural numbers)

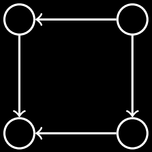
Assume the domain is the set of natural numbers, and let Dxy mean x divides y.

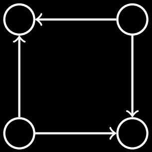
- Use $\varphi_{=0}(x)$ to refer to the formula $\forall z D z x$
- Use $\varphi_{=1}(x)$ to refer to the formula $\forall z D x z$
- Consider the formula $\varphi_P(x) = \forall y(Dyx \rightarrow (\varphi_{=1}(y) \vee y = x))$

In this sense, 0, 1 in natural numbers and the predicate "prime" are definable from the "divisibility" relation.

"Cases" or "Pictures" for Predicate Logic

Write a formula that is true for the graph on the left and false for the graph on the right.





Homework Note

Conventions for submitting predicate logic reasonings

- If it can be formalized in propositional logic, try to submit it in propositional logic.
- When submitting a predicate logic reasoning, select "First-Order" for Logic type.
- In the translation part, you need to specify the translations. For example, translating Lxy as "x is greater than y" (which also implies L is a binary predicate), and translating a constant like a as "Adam".

Exercises & Discussion

Assuming the domain is the characters in the universe of "The Three-Body Problem", we use l to denote Luo Ji, z to denote Zhuang Yan, L to represent the relation "loves", and R to represent the relation "respects". Translate the following statements:

- Luo Ji is not loved by everyone.
- Luo Ji and Zhuang Yan respect each other.
- Zhuang Yan respects everyone who loves Luo Ji.

Exercises & Discussion

Define your own domain and symbols to translate the following statements:

- A barking dog doesn't bite.
- A friend of Adam's friend is also his friend.
- There exists a smallest natural number.
- (*) You are not a fish, how do you know the joy of a fish?

Exercises & Discussion

Assuming the domain is humans, translate the following statements:

- Every boy loves Alice.
- Not every boy loves himself.
- Alice loves a boy who loves Carol.
- A boy who loves Carol is loved by no boy or girl.

Can you draw a graph such that your translations for all the above statements are true?

Exercises & Discussion

Is the divisibility relation on natural numbers:

- Transitive?
- Reflexive?
- Symmetric?
- Linear?

Exercises & Discussion

Consider the following graph, we use Rxy to denote that there is an arrow from node x to y.

- Is $\exists x \forall y Rxy$ true for the above graph?
- If it is true, why? If it is not true, try to make the minimal change to the above graph to make this formula true.

Exercises & Discussion *

Use graphs to represent social networks, where Rxy means x can contact y. We call a graph connected if and only if there is at least one arrow between any two vertices in the graph.

- Write a predicate logic formula such that any graph satisfying this formula is connected.
- We call a vertex in a graph a Great Communicator (GC) if and only if it can contact any vertex in the graph through at most two arrows. Prove: Any finite, reflexive, connected graph has a GC. What about infinite graphs?

Exercises & Discussion

Translate the following daily language sentences into predicate logic formulas:

- If Adam loves Bob, then Bob also loves Adam.
- Adam and Bob love each other.
- Adam and Bob do not love each other.
- If Adam and Charles love Bob, then neither Bob nor Charles loves Adam.

Exercises & Discussion

Reformulate the following formulas using only the predicates $<$ (less than) and $=$ (equal to):

- "$x \leq y \wedge y \leq z$"
- "$\neg(x \leq y \wedge y \leq z)$"