Mathematical Logic

Lecture 5: Metatheory of Propositional Logic

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Metatheory: About Proof Systems

- Different formal proof systems for Propositional Logic:
  1. The Hilbert-style Axiom System (H): Minimal rules, many axioms.
  2. Natural Deduction (ND): Many rules, no axioms, closely mirrors human reasoning.
- We will prove Metatheorem about the proof systems.
- We will show they are essentially equivalent.

Notation

To distinguish between provability in the two systems, we will use subscripts on the $\vdash$ symbol:

- $\Sigma \vdash_H \varphi$: φ is provable from premises Σ in the Hilbert-style Axiom System.
- $\Sigma \vdash_{ND} \varphi$: φ is provable from premises Σ in Natural Deduction.

Formal Proofs as Inductive Structures

To prove theorems *about* our proof systems, we often use induction on the structure of the proof.

Hilbert-style Proof (Sequence) : A sequence of formulas $\varphi_1, \varphi_2, \ldots, \varphi_n$ where each φ_i is either:

- An axiom (P1, P2, P3).
- A premise from Σ.
- Derived from two previous formulas φ_j, φ_k $(j, k < i)$ using Modus Ponens.

Formal Proofs as Inductive Structures

Natural Deduction Proof (Tree) : A tree where nodes are formulas, and the root is the conclusion φ. Each step in the tree is either:

- An assumption (a leaf node).
- Formed by applying an ND introduction/elimination rule to smaller proof trees.

Simulating ND in H

- Goal: Prove $\Sigma \vdash_{ND} \varphi \implies \Sigma \vdash_{H} \varphi$.
- Strategy: Induction on the structure of the ND proof tree. Show that H can simulate every rule of ND.
- For rules without discharged assumptions (e.g., $\rightarrow E$, which is just Modus Ponens), this is trivial.
- For rules that discharge assumptions (like $\rightarrow I$), we need a powerful metalogical tool: the Deduction Theorem .

The Deduction Theorem

Theorem (Deduction Theorem for the Hilber-style proof system)

If $\Sigma \cup \{\varphi\} \vdash_H \psi$, then $\Sigma \vdash_H \varphi \to \psi$.

- This perfectly mirrors the $\to I$ rule in ND!
- If we can prove ψ by temporarily assuming φ, we can prove the conditional $\varphi \to \psi$ without that assumption.

Simulating Other Connectives

- H only uses $\neg$ and $\rightarrow$. To simulate ND rules for $\wedge$ and $\vee$, we treat them as abbreviations in H:
  - $\varphi \vee \psi \quad \equiv \quad \neg\varphi \rightarrow \psi$
  - $\varphi \wedge \psi \quad \equiv \quad \neg(\varphi \rightarrow \neg\psi)$

Simulating Other Connectives

- Simulating an ND rule means proving the corresponding theorem in H.
- Example ($\vee I$): From φ, infer $\varphi \vee \psi$.
  - In H, this means proving $\varphi \vdash_H \neg\varphi \to \psi$.
  - By the Deduction Theorem, this is $\vdash_H \varphi \to (\neg\varphi \to \psi)$, which is logically equivalent to the known theorem $\vdash_H \neg\varphi \to (\varphi \to \psi)$.

Mechanical Translation: An Example

Let's translate an ND proof into an H proof sequence to see the algorithm in action.

Example (Hypothetical Syllogism)

Goal: Show $\{p \to q, q \to r\} \vdash_H p \to r$

Step 1: The ND Proof

$$\dfrac{\dfrac{\dfrac{[p]^1 \qquad p \to q}{q} \to E \qquad q \to r}{r} \to E}{p \to r} \to I_1$$

Mechanical Translation: An Example

Step 2: H proof with the assumption as a premise

Before the $\rightarrow I$ step, the ND proof derives r from the premises plus the assumption p. This corresponds to a standard H proof sequence for $\{p \rightarrow q, q \rightarrow r, p\} \vdash_H r$:

1. $p \rightarrow q$ (Premise)
2. p (Premise / Assumption)
3. q (MP 1, 2)
4. $q \rightarrow r$ (Premise)
5. r (MP 3, 4)

Mechanical Translation: An Example

Step 3: Applying the Deduction Theorem Algorithm

To discharge p, the Deduction Theorem says we must prefix every line φ_i in the previous proof with "$p \rightarrow$":

1. $p \rightarrow q$ (Premise)
2. $(p \rightarrow q) \rightarrow (p \rightarrow (p \rightarrow q))$ (P1)
3. $p \rightarrow (p \rightarrow q)$ (MP 1, 2)
4. $p \rightarrow p$ (Known Theorem)
5. $(p \rightarrow (p \rightarrow q)) \rightarrow ((p \rightarrow p) \rightarrow (p \rightarrow q))$ (P2)
6. $(p \rightarrow p) \rightarrow (p \rightarrow q)$ (MP 3, 5)

Mechanical Translation: An Example

Step 3 (Continued):

7	$p \to q$	(MP 4, 6)
8	$q \to r$	(Premise)
9	$(q \to r) \to (p \to (q \to r))$	(P1)
10	$p \to (q \to r)$	(MP 8, 9)
11	$(p \to (q \to r)) \to ((p \to q) \to (p \to r))$	(P2)
12	$(p \to q) \to (p \to r)$	(MP 10, 11)
13	$p \to r$	(MP 7, 12)

Simulating H in ND

- Goal: Prove $\Sigma \vdash_H \varphi \implies \Sigma \vdash_{ND} \varphi$.
- Strategy: Induction on the length of the proof sequence in H.
- Inductive Step: If φ is derived from ψ and $\psi \to \varphi$ by Modus Ponens, then by the inductive hypothesis, we already have ND proof trees for ψ and $\psi \to \varphi$. We simply combine them using $\to E$:

Simulating H in ND

$$\dfrac{\begin{array}{cc}\vdots & \vdots \\ \psi & \psi \to \varphi\end{array}}{\varphi} \to E$$

- Base Cases: We must show that every axiom of H is a theorem in ND.

Proving Axiom P1 in ND

Axiom P1: $\varphi \to (\psi \to \varphi)$

$$\dfrac{\dfrac{[\varphi]^1}{\psi \to \varphi} \to I_2}{\varphi \to (\psi \to \varphi)} \to I_1$$

Proving Axiom P2 in ND

Axiom P2: $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$

$$\dfrac{\dfrac{\dfrac{[\varphi \to (\psi \to \chi)]^3 \quad [\varphi]^1}{\psi \to \chi} \to E \quad \dfrac{[\varphi \to \psi]^2 \quad [\varphi]^1}{\psi} \to E}{\chi} \to E}{\varphi \to \chi} \to I_1}{(\varphi \to \psi) \to (\varphi \to \chi)} \to I_2}{(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))} \to I_3$$

Proving Axiom P3 in ND

Axiom P3: $(\neg\varphi \to \psi) \to (\neg\varphi \to \neg\psi) \to \varphi$

$$\dfrac{\dfrac{\dfrac{[\neg\varphi \to \psi]^3 \quad [\neg\varphi]^1}{\psi}\to E \quad \dfrac{[\neg\varphi \to \neg\psi]^2 \quad [\neg\varphi]^1}{\neg\psi}\to E}{\varphi}\neg \text{ Elim } (\text{RAA})_1}{(\neg\varphi \to \neg\psi) \to \varphi}\to I_2}{(\neg\varphi \to \psi) \to (\neg\varphi \to \neg\psi) \to \varphi}\to I_3$$

Strong Completeness

- We can show that our proof systems are Sound : they never prove falsehoods. If $\Sigma \vdash \varphi$, then $\Sigma \vDash \varphi$.
- Are they powerful enough deliver all expected proofs?
- Completeness Theorem: If $\Sigma \vDash \varphi$, then $\Sigma \vdash_H \varphi$.
- This establishes a perfect equivalence between syntax (symbol manipulation $\vdash$) and semantics (truth $\vDash$).

Kalmár's Lemma

The core of the completeness proof is Kalmár's Lemma , which links truth valuations to formal provability.

Notation: For a valuation V and a formula φ:

$$\varphi^V = \begin{cases} \varphi & \text{if } V(\varphi) = 1 \\ \neg\varphi & \text{if } V(\varphi) = 0 \end{cases}$$

Let $L(V)$ be the set of p^V for all variables p in a formula φ.

Kalmár's Lemma

Lemma

For any formula φ and valuation V:

$$L(V) \vdash_H \varphi^V$$

Kalmár's Lemma

Example: Let $\varphi = (p \to q)$. Suppose $V(p) = 1$ and $V(q) = 0$.

- Here, $p^V = p$, $q^V = \neg q$. So $L(V) = \{p, \neg q\}$.
- Since $V(\varphi) = 0$, we have $\varphi^V = \neg(p \to q)$.
- Kalmár's Lemma claims: $\{p, \neg q\} \vdash_H \neg(p \to q)$.

Kalmár's Lemma

Proof.

by induction on the structure of φ:

Base Case: φ is a propositional variable p. Then $L(V) = \{p^V\}$, and trivially $p^V \vdash_H p^V$.

Inductive Step ($\neg$): Suppose $\varphi = \neg\psi$. If $V(\psi) = 1$, then by IH $L(V) \vdash_H \psi$. We need to show $L(V) \vdash_H (\neg\psi)^V$, which is $\neg\neg\psi$. Since $\vdash_H \psi \to \neg\neg\psi$, we have $L(V) \vdash_H \neg\neg\psi$. If $V(\psi) = 0$, then $L(V) \vdash_H \neg\psi$. We need $L(V) \vdash_H (\neg\psi)^V$, which is $\neg\psi$. This is trivial.

Kalmár's Lemma

Proof.

Inductive Step ($\rightarrow$): Suppose $\varphi = \psi \rightarrow \chi$.

- If $V(\chi) = 1$: Then $V(\varphi) = 1$. By IH, $L(V) \vdash_H \chi$. We need $L(V) \vdash_H \psi \rightarrow \chi$. By Axiom P1, $\vdash_H \chi \rightarrow (\psi \rightarrow \chi)$. Using MP, $L(V) \vdash_H \psi \rightarrow \chi$.
- If $V(\psi) = 0$: Then $V(\varphi) = 1$. By IH, $L(V) \vdash_H \neg\psi$. We need $L(V) \vdash_H \psi \rightarrow \chi$. Using the known theorem $\vdash_H \neg\psi \rightarrow (\psi \rightarrow \chi)$ and MP, we get $L(V) \vdash_H \psi \rightarrow \chi$.

Kalmár's Lemma

Proof.

- If $V(\psi) = 1$ and $V(\chi) = 0$: Then $V(\varphi) = 0$. By IH, $L(V) \vdash_H \psi$ and $L(V) \vdash_H \neg\chi$. We need $L(V) \vdash_H \neg(\psi \to \chi)$. Using the known theorem $\vdash_H \psi \to (\neg\chi \to \neg(\psi \to \chi))$ and MP twice, we get $L(V) \vdash_H \neg(\psi \to \chi)$.

The Elimination Step

If φ is a tautology ($\vDash \varphi$), then $V(\varphi) = 1$ for every possible valuation V. Thus Kalmár's Lemma gives us:

$$\{p_1^V, p_2^V, \ldots, p_n^V\} \vdash_H \varphi \quad \text{for all } 2^n \text{ valuations } V$$

The Elimination Step

We can eliminate the assumptions one by one:

- We have $\{p_1\} \cup \Sigma \vdash_H \varphi$ and $\{\neg p_1\} \cup \Sigma \vdash_H \varphi$.
- By the Deduction Theorem: $\Sigma \vdash_H p_1 \to \varphi$ and $\Sigma \vdash_H \neg p_1 \to \varphi$.
- Using the H theorem $\vdash_H (p_1 \to \varphi) \to ((\neg p_1 \to \varphi) \to \varphi)$, we get:
- $\Sigma \vdash_H \varphi$. (We eliminated p_1!)

Repeating this for all variables leaves us with $\emptyset \vdash_H \varphi$.

From Weak to Strong Completeness

We just proved Weak Completeness: $\vDash \varphi \implies \vdash_H \varphi$.

How do we get Strong Completeness: $\Sigma \vDash \varphi \implies \Sigma \vdash_H \varphi$?

- If $\Sigma \vDash \varphi$, then by the Compactness Theorem for propositional logic, there is a finite subset $\{\sigma_1, \ldots, \sigma_k\} \subseteq \Sigma$ such that $\{\sigma_1, \ldots, \sigma_k\} \vDash \varphi$.
- Then the formula $\sigma_1 \to \sigma_2 \to \cdots \to \sigma_k \to \varphi$ is a tautology.

From Weak to Strong Completeness

- By Weak Completeness, it is provable in H:

  $\vdash_H \sigma_1 \to (\cdots \to \varphi)$.

- Since each $\sigma_i \in \Sigma$, we can use Modus Ponens k times to get $\Sigma \vdash_H \varphi$.

The Compactness Theorem

Theorem

A set of propositional formulas Σ is satisfiable if and only if every finite subset of Σ is satisfiable (Σ is finitely satisfiable).

- ($\Rightarrow$) is trivial.
- ($\Leftarrow$) If every finite subset of Σ can be satisfied (perhaps by completely different valuations), then the entire infinite set Σ can be satisfied simultaneously by a single valuation.

Proof of Compactness

Proof Idea: We build a satisfying valuation V variable by variable $(p_1, p_2, \dots)$.

Step 1: Can we assign a value to p_1?

- We know Σ is finitely satisfiable. Then, at least one of $\Sigma \cup \{p_1\}$ or $\Sigma \cup \{\neg p_1\}$ be finitely satisfiable:

  Suppose neither is. Then there exist finite subsets $\Delta_1, \Delta_2 \subseteq \Sigma$ such that $\Delta_1 \cup \{p_1\}$ is unsatisfiable and $\Delta_2 \cup \{\neg p_1\}$ is unsatisfiable.

Proof of Compactness

Proof Idea: We build a satisfying valuation V variable by variable $(p_1, p_2, \dots)$.

Step 1: Can we assign a value to p_1?

- Let $\Delta = \Delta_1 \cup \Delta_2$. Δ is a finite subset of Σ.
- Any valuation satisfying Δ must make p_1 false (due to Δ_1) AND true (due to Δ_2). Impossible!
- Thus, Δ is unsatisfiable, contradicting that Σ is finitely satisfiable.

Proof of Compactness

Step 2: Inductive Construction

- We choose $V(p_1) = v_1$ such that $\Sigma \cup \{p_1^{v_1}\}$ is finitely satisfiable. (Letting $p^0 = \neg p$, and $p^1 = p$)
- Inductively, having chosen $V(p_1), \ldots, V(p_n)$, we can pick $V(p_{n+1}) = v_{n+1}$ such that $\Sigma \cup \{p_1^{v_1}, \ldots, p_{n+1}^{v_{n+1}}\}$ is finitely satisfiable.
- This process defines a complete valuation V for all variables.

Proof of Compactness

Step 3: Verification

- Does V satisfy Σ? Let $\varphi \in \Sigma$.
- φ only contains finitely many variables, say among $p_1, \ldots, p_k$.
- By construction, $\Sigma \cup \{p_1^{v_1}, \ldots, p_k^{v_k}\}$ is finitely satisfiable, which means $\{\varphi, p_1^{v_1}, \ldots, p_k^{v_k}\}$ is satisfiable.
- But V is the **only** valuation that satisfies $\{p_1^{v_1}, \ldots, p_k^{v_k}\}$ for these variables! Thus, V must satisfy φ. ■

Exercise & Discussion

- Prove the following are equivalent:
  - The Compactness Theorem
  - If $\Sigma \vDash \varphi$, then there is a finite subset $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \vDash \varphi$.
- In the Step 2 of the proof of Compactness Theorem, what if both $\Sigma \cup \{p_i^0\}$ and $\Sigma \cup \{p_i^1\}$ are finitely satisfiable?

Exercise & Discussion

- Use induction on the structure of formulas to prove the Agreement Theorem : If two valuations V_1 and V_2 agree on all atomic formula occurring in φ, then $V_1(\varphi) = V_2(\varphi)$.
- Use induction on the length of proof sequences to prove the Soundness Theorem for our Hilbert system: $\Sigma \vdash_H \varphi \implies \Sigma \vDash \varphi$.

Exercise & Discussion

Cell Tower Frequencies:

A telecom company is deploying a vast (modeled as infinite) network of cell towers T. They have exactly 3 frequency bands. To avoid interference, no two adjacent towers can broadcast on the same frequency.

Exercise & Discussion

- Suppose engineering verifies that for any **finite** sub-network, there is a valid frequency assignment.
- Define a set of propositional variables (e.g., $b_{t,i}$ means "tower t uses frequency band i").
- Write down an infinite set of formulas Σ expressing "the entire network T has a valid assignment".
- Use the Compactness Theorem to rigorously prove that the entire network can operate without interference.

Exercise & Discussion

A set of formulas Σ is inconsistent if there is some formula φ such that $\Sigma \vdash_H \varphi$ and $\Sigma \vdash_H \neg\varphi$. Otherwise, Σ is consistent .

- Use the Soundness and Strong Completeness theorems to prove that: Σ is consistent $\iff$ Σ is satisfiable.
- Prove that if $\Sigma \cup \{\neg\varphi\}$ is inconsistent, then $\Sigma \vdash_H \varphi$.
- Without using the Compactness Theorem, prove that if Σ is inconsistent, then there exists a finite subset $\Sigma_0 \subseteq \Sigma$ that is also inconsistent.