Mathematical Logic

Lecture 4: Propositional Logic - Natural Deduction

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Proofs

The word ”Proof” in Everyday Language

- He tried to prove his innocence with an alibi.
- The detected gravitational waves proved Einstein’s theory.
- How are mathematical proofs similar to or different from these?

Proofs

Mathematics is a game played according to certain simple rules with meaningless marks on paper.

David Hilbert

Proofs

Example

Euclid's Elements

- Postulate 1: To draw a straight line from any point to any point.
- Postulate 3: To describe a circle with any center and distance.
- Proposition 1: To construct an equilateral triangle on a given finite straight line.

Proposition 1: To construct an equilateral triangle on a given finite straight line.

Proof.

Let AB be the given finite straight line.

(By Postulate 3) Describe the circle with center A and radius AB, and the circle with center B and radius AB;

Let C be a point where the two circles intersect;

(By Postulate 1) Draw straight lines AC and BC;

(By Definition of a circle) The length of $AC = AB$, and the length of $BC = AB$.

Proposition 1: To construct an equilateral triangle on a given finite straight line.

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Proofs

Definition

- A proof is a sequence (or a tree) of statements (or formulas), where each statement is either an axiom, an assumption, or derived from previous statements using specific deduction rules.
- A statement is a theorem if and only if it appears at the end of a proof without any undischarged assumptions.
- A set of axioms and deduction rules constitutes a formal proof system.

A Hilbert-style Proof System

Definition (A Hilbert-style Propositional Logic Axiom System)

- Axioms:

  (P1) $\varphi \to \psi \to \varphi$

  (P2) $(\varphi \to \psi \to \chi) \to (\varphi \to \psi) \to (\varphi \to \chi)$

  (P3) $(\neg\varphi \to \psi) \to (\neg\varphi \to \neg\psi) \to \varphi$

- Deduction Rule: If φ and $(\varphi \to \psi)$ are theorems, then ψ is also a theorem. (Modus Ponens)

A Hilbert-style Proof System

Note:

- We use meta-language symbols φ, ψ, χ to describe axioms and theorems. This implies that this axiom system actually contains an infinite number of axioms.
- This specific axiom system only uses two logical connectives: $\neg$ and $\rightarrow$.

A Hilbert-style Proof System

Example (A Proof in the Axiom System)

We want to show that all formulas of the form $\varphi \to \varphi$ are theorems in this system.

1. $\varphi \to (\psi \to \varphi) \to \varphi$ (P1)
2. $(\varphi \to (\psi \to \varphi) \to \varphi) \to (\varphi \to (\psi \to \varphi)) \to (\varphi \to \varphi)$ (P2)
3. $(\varphi \to \psi \to \varphi) \to (\varphi \to \varphi)$ Modus Ponens (1, 2)
4. $\varphi \to \psi \to \varphi$ (P1)
5. $\varphi \to \varphi$ Modus Ponens (3, 4)

A Hilbert-style Proof System

Example $(\neg\neg\varphi \to \varphi)$

1. $\neg\varphi \to \neg\varphi$
2. $(\neg\varphi \to \neg\varphi) \to (\neg\varphi \to \neg\neg\varphi) \to \varphi$ (P3)
3. $(\neg\varphi \to \neg\neg\varphi) \to \varphi$
4. $((\neg\varphi \to \neg\neg\varphi) \to \varphi) \to \neg\neg\varphi \to ((\neg\varphi \to \neg\neg\varphi) \to \varphi)$

A Hilbert-style Proof System

Example $(\neg\neg\varphi \to \varphi)$

5. $\neg\neg\varphi \to ((\neg\varphi \to \neg\neg\varphi) \to \varphi)$
6. $\big(\neg\neg\varphi \to ((\neg\varphi \to \neg\neg\varphi) \to \varphi)\big) \to (\neg\neg\varphi \to \neg\varphi \to \neg\neg\varphi) \to (\neg\neg\varphi \to \varphi)$
7. $(\neg\neg\varphi \to \neg\varphi \to \neg\neg\varphi) \to \neg\neg\varphi \to \varphi$
8. $\neg\neg\varphi \to \neg\varphi \to \neg\neg\varphi$
9. $\neg\neg\varphi \to \varphi$

A Hilbert-style Proof System

Example $((a \to b \to c) \to b \to a \to c)$

We only provide a proof outline here:

1. $\big((a \to b \to c) \to (((a \to b) \to (a \to c)) \to (b \to ((a \to b) \to (a \to c))))\big) \to ((a \to b \to c) \to ((a \to b) \to (a \to c))) \to (a \to b \to c) \to (b \to ((a \to b) \to (a \to c)))$ (P2)
2. $((a \to b) \to (a \to c)) \to (b \to ((a \to b) \to (a \to c)))$ (P1)
3. $(a \to b \to c) \to (((a \to b) \to (a \to c)) \to (b \to ((a \to b) \to (a \to c))))$ (P1) MP (2)

A Hilbert-style Proof System

Example $((a \to b \to c) \to b \to a \to c)$

4. $(a \to b \to c) \to ((a \to b) \to (a \to c))$ (P2)
5. $(a \to b \to c) \to (b \to ((a \to b) \to (a \to c)))$ (1) MP (3), (4)
6. $(b \to (a \to b) \to (a \to c)) \to (b \to a \to b) \to (b \to a \to c)$(P2)
7. (6) $\to (a \to b \to c) \to$ (6) (P1)
8. $(a \to b \to c) \to$ (6) (P1) MP (6)

A Hilbert-style Proof System

Example $((a \to b \to c) \to b \to a \to c)$

9 $(a \to b \to c) \to (b \to a \to b) \to (b \to a \to c)$ (P2) MP (8), (5)

10 $(a \to b \to c) \to b \to (a \to b)$ (P1) MP (P1)

11 $(a \to b \to c) \to b \to a \to c$ (P2) MP (5), (6)

A Hilbert-style Proof System

Example $((a \to b) \to (b \to c) \to (a \to c))$

Proof outline:

1. $(a \to b \to c) \to (a \to b) \to a \to c$ (P2)
2. $((a \to b \to c) \to (a \to b) \to a \to c) \to (b \to c) \to ((a \to b \to c) \to (a \to b) \to a \to c)$ (P1)
3. $(b \to c) \to ((a \to b \to c) \to (a \to b) \to a \to c)$ (2) MP (1)

A Hilbert-style Proof System

Example $((a \to b) \to (b \to c) \to (a \to c))$

4. $(b \to c) \to (a \to b \to c)$ (P1)
5. $(b \to c) \to (a \to b) \to (a \to c)$ (P2) MP (3), (4)
6. $((b \to c) \to (a \to b) \to (a \to c)) \to ((a \to b) \to (b \to c) \to (a \to c))$ From prev example
7. $(a \to b) \to (b \to c) \to (a \to c)$ (6) MP (5)

A Hilbert-style Proof System

The following are also theorems (Exercises):

- $(\varphi \to \psi) \to (\varphi \to \neg\psi) \to \neg\varphi$
- $\varphi \to \neg\neg\varphi$
- $\neg\varphi \to \varphi \to \psi$
- $(\varphi \to \psi) \to (\neg\varphi \to \psi) \to \psi$
- $\varphi \to \neg\psi \to \neg(\varphi \to \psi)$

A Hilbert-style Proof System

In reality, we often prove conclusions from some given premises in addition to the axioms.

Example (Party Planning)

Suppose we are planning a party, but with the following constraints:

- If Alice or Bob comes, then Charlie must come.
- If Alice does not come, Bob will come.
- If Bob comes, then Charlie will not come.

We can use a truth table to find a solution.

A Hilbert-style Proof System

In reality, we often prove conclusions from some given premises in addition to the axioms.

Example (Party Planning)

Suppose we are planning a party, but with the following constraints:

- $(A \vee B) \rightarrow C$
- If Alice does not come, Bob will come.
- If Bob comes, then Charlie will not come.

We can use a truth table to find a solution.

A Hilbert-style Proof System

In reality, we often prove conclusions from some given premises in addition to the axioms.

Example (Party Planning)

Suppose we are planning a party, but with the following constraints:

- $(A \vee B) \rightarrow C$
- $\neg A \rightarrow B$
- If Bob comes, then Charlie will not come.

We can use a truth table to find a solution.

A Hilbert-style Proof System

In reality, we often prove conclusions from some given premises in addition to the axioms.

Example (Party Planning)

Suppose we are planning a party, but with the following constraints:

- $(A \vee B) \to C$
- $\neg A \to B$
- $B \to \neg C$

We can use a truth table to find a solution.

A Hilbert-style Proof System

In reality, we often prove conclusions from some given premises in addition to the axioms.

Example (Party Planning)

Suppose we are planning a party, but with the following constraints:

- $(A \vee B) \rightarrow C$
- $\neg A \rightarrow B$
- $B \rightarrow \neg C$

We can use a truth table to find a solution.

A Hilbert-style Proof System

In reality, we often prove conclusions from some given premises in addition to the axioms.

Example (Party Planning)

Suppose we are planning a party, but with the following constraints:

- $A \to C$, $B \to C$
- $\neg A \to B$
- $B \to \neg C$

We can use a truth table to find a solution.

A Hilbert-style Proof System

We can also figure it out via proofs:

- $(B \to C) \to (B \to \neg C) \to \neg B$
- $B \to C$
- $(B \to \neg C) \to \neg B$
- $B \to \neg C$
- $\neg B$

A Hilbert-style Proof System

Notation

Let Σ be a set of formulas. If there exists a sequence of formulas where each formula is either an axiom, a formula in Σ, or derived from previous formulas using deduction rules, we say the formula at the end of the sequence is provable from Σ . If ψ is such a formula, we write: $\Sigma \vdash \psi$. When $\Sigma = \{\varphi_1, \ldots, \varphi_n\}$ is finite, we write $\varphi_1, \ldots, \varphi_n \vdash \psi$. When Σ is empty, we write $\vdash \psi$, meaning ψ is a theorem of the axiom system.

A Hilbert-style Proof System

Example

- $A \to C, B \to C, \neg A \to B, B \to \neg C \vdash \neg B$
- $A \to C, B \to C, \neg A \to B, B \to \neg C \vdash A$
- $A \to C, B \to C, \neg A \to B, B \to \neg C \vdash C$

Hilbert-style Systems vs. Natural Deduction

- Hilbert-style Axiom System have many axioms but very few deduction rules (often just Modus Ponens). A proof in Hilbert-style Axiom System is linear.
  - Pros: Easy to prove meta-theorems about the system.
  - Cons: Extremely difficult and unintuitive for humans to construct formal proofs (internal theorems).

Hilbert-style Systems vs. Natural Deduction

- Natural Deduction is a proof system designed to closely mimic human mathematical reasoning.
  - It usually has no axioms, but provides intuitive Introduction and Elimination rules for each logical connective.
  - A proof in Natural Deduction is a tree.

Natural Deduction Rules

Rules for $\wedge$ (Conjunction):

$$\dfrac{\begin{array}{c}\vdots \\ \varphi\end{array} \qquad \begin{array}{c}\vdots \\ \psi\end{array}}{\varphi \wedge \psi} \; \wedge \text{ Intro}$$

$\Gamma \vdash \varphi$ and $\Gamma \vdash \psi \Rightarrow \Gamma \vdash \varphi \wedge \psi$

Natural Deduction Rules

Rules for $\wedge$ (Conjunction):

$$\dfrac{\begin{matrix}\vdots & & \vdots \\ \varphi & & \psi\end{matrix}}{\varphi \wedge \psi} \wedge \text{ Intro}$$

$$\Gamma \vdash \varphi \text{ and } \Gamma \vdash \psi \Rightarrow \Gamma \vdash \varphi \wedge \psi$$

Natural Deduction Rules

Rules for $\wedge$ (Conjunction):

$$\dfrac{\begin{array}{c}\vdots\\ \varphi \wedge \psi\end{array}}{\varphi}\ \wedge \text{ Elim (Left)}$$

$$\Gamma \vdash \varphi \wedge \psi \Rightarrow \Gamma \vdash \varphi$$

$$\dfrac{\begin{array}{c}\vdots\\ \varphi \wedge \psi\end{array}}{\psi}\ \wedge \text{ Elim (Right)}$$

$$\Gamma \vdash \varphi \wedge \psi \Rightarrow \Gamma \vdash \psi$$

Natural Deduction Rules

Rules for $\wedge$ (Conjunction):

$$\frac{\begin{array}{c}\vdots\\ \varphi \wedge \psi\end{array}}{\varphi}\ \wedge\ \text{Elim (Left)}$$

$$\Gamma \vdash \varphi \wedge \psi \Rightarrow \Gamma \vdash \varphi$$

$$\frac{\begin{array}{c}\vdots\\ \varphi \wedge \psi\end{array}}{\psi}\ \wedge\ \text{Elim (Right)}$$

$$\Gamma \vdash \varphi \wedge \psi \Rightarrow \Gamma \vdash \psi$$

Natural Deduction Rules

Rules for $\vee$ (Disjunction):

$$\frac{\begin{matrix}\vdots \\ \varphi\end{matrix}}{\varphi \vee \psi}\ \vee\ \text{Intro (Left)} \qquad\qquad \frac{\begin{matrix}\vdots \\ \psi\end{matrix}}{\varphi \vee \psi}\ \vee\ \text{Intro (Right)}$$

$$\Gamma \vdash \varphi \Rightarrow \Gamma \vdash \varphi \vee \psi \qquad\qquad \Gamma \vdash \psi \Rightarrow \Gamma \vdash \varphi \vee \psi$$

Natural Deduction Rules

Rules for $\vee$ (Disjunction):

$$\dfrac{\begin{array}{c}\vdots\\ \varphi\end{array}}{\varphi \vee \psi} \vee \text{ Intro (Left)} \qquad \dfrac{\begin{array}{c}\vdots\\ \psi\end{array}}{\varphi \vee \psi} \vee \text{ Intro (Right)}$$

$$\Gamma \vdash \varphi \Rightarrow \Gamma \vdash \varphi \vee \psi \qquad \Gamma \vdash \psi \Rightarrow \Gamma \vdash \varphi \vee \psi$$

Natural Deduction Rules

Rules for $\vee$ (Disjunction):

$$\dfrac{\begin{array}{c}\vdots \\ \varphi \vee \psi\end{array} \qquad \begin{array}{c}[\varphi] \\ \vdots \\ \chi\end{array} \qquad \begin{array}{c}[\psi] \\ \vdots \\ \chi\end{array}}{\chi} \ \vee \text{ Elim}$$

$\Gamma \vdash \varphi \vee \psi$, and $\Gamma, \varphi \vdash \chi$, and $\Gamma, \psi \vdash \chi \Rightarrow \Gamma \vdash \chi$

Natural Deduction Rules

Rules for $\vee$ (Disjunction):

$$\dfrac{\begin{matrix} \\ \vdots \\ \varphi \vee \psi \end{matrix} \qquad \begin{matrix} [\varphi] \\ \vdots \\ \chi \end{matrix} \qquad \begin{matrix} [\psi] \\ \vdots \\ \chi \end{matrix}}{\chi} \vee \text{ Elim}$$

$\Gamma \vdash \varphi \vee \psi$, and $\Gamma, \varphi \vdash \chi$, and $\Gamma, \psi \vdash \chi \Rightarrow \Gamma \vdash \chi$

Natural Deduction Rules

Rules for $\to$ (Implication):

$$\dfrac{\begin{array}{c}[\varphi]\\ \vdots \\ \psi\end{array}}{\varphi \to \psi} \to \text{Intro} \qquad\qquad \dfrac{\begin{array}{c}\vdots \\ \varphi\end{array} \quad \begin{array}{c}\vdots \\ \varphi \to \psi\end{array}}{\psi} \to \text{Elim (MP)}$$

$\Gamma, \varphi \vdash \psi \Rightarrow \Gamma \vdash \varphi \to \psi$ $\qquad$ $\Gamma \vdash \varphi$ and $\Gamma \vdash \varphi \to \psi \Rightarrow \Gamma \vdash \psi$

Natural Deduction Rules

Rules for $\rightarrow$ (Implication):

$$\dfrac{\begin{matrix}[\varphi] \\ \vdots \\ \psi\end{matrix}}{\varphi \rightarrow \psi} \rightarrow \text{Intro} \qquad \dfrac{\begin{matrix}\vdots \\ \varphi\end{matrix} \qquad \begin{matrix}\vdots \\ \varphi \rightarrow \psi\end{matrix}}{\psi} \rightarrow \text{Elim (MP)}$$

$\Gamma, \varphi \vdash \psi \Rightarrow \Gamma \vdash \varphi \rightarrow \psi$

$\Gamma \vdash \varphi$ and $\Gamma \vdash \varphi \rightarrow \psi \Rightarrow \Gamma \vdash \psi$

Natural Deduction Rules

Rules for $\neg$ (Negation):

$$\dfrac{\begin{matrix}[\varphi] & \quad & [\varphi] \\ \vdots & & \vdots \\ \psi & & \neg\psi\end{matrix}}{\neg\varphi}\ \neg\ \text{Intro} \qquad\qquad \dfrac{\begin{matrix}[\neg\varphi] & \quad & [\neg\varphi] \\ \vdots & & \vdots \\ \psi & & \neg\psi\end{matrix}}{\varphi}\ \neg\ \text{Elim (RAA)}$$

$\Gamma, \varphi \vdash \psi$ and $\Gamma, \varphi \vdash \neg\psi \Rightarrow \Gamma \vdash \neg\varphi$

$\Gamma, \neg\varphi \vdash \psi$ and $\Gamma, \neg\varphi \vdash \neg\psi \Rightarrow \Gamma \vdash \varphi$

Natural Deduction Rules

Rules for $\neg$ (Negation):

$$\dfrac{\begin{matrix}[\varphi] & & [\varphi] \\ \vdots & & \vdots \\ \psi & & \neg\psi\end{matrix}}{\neg\varphi}\ \neg\ \text{Intro}$$

$$\dfrac{\begin{matrix}[\neg\varphi] & & [\neg\varphi] \\ \vdots & & \vdots \\ \psi & & \neg\psi\end{matrix}}{\varphi}\ \neg\ \text{Elim (RAA)}$$

$\Gamma, \varphi \vdash \psi$ and $\Gamma, \varphi \vdash \neg\psi \Rightarrow \Gamma \vdash \neg\varphi$

$\Gamma, \neg\varphi \vdash \psi$ and $\Gamma, \neg\varphi \vdash \neg\psi \Rightarrow \Gamma \vdash \varphi$

Example 1: Commutativity of $\wedge$

Goal: Show that $p \wedge q \vdash q \wedge p$

$$\begin{array}{c} ? \\ \vdots \\ q \wedge p \end{array}$$

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Goal: Show that $p \wedge q \vdash q \wedge p$

$$\dfrac{\begin{array}{c} ? \\ \vdots \\ q \end{array} \qquad \begin{array}{c} ? \\ \vdots \\ p \end{array}}{q \wedge p} \; \wedge \text{ Intro}$$

Example 1: Commutativity of $\wedge$

Goal: Show that $p \wedge q \vdash q \wedge p$

$$\dfrac{\dfrac{p \wedge q}{q}\ \wedge E_R \qquad \dfrac{p \wedge q}{p}\ \wedge E_L}{q \wedge p}\ \wedge\ \text{Intro}$$

Example 2: Hypothetical Syllogism

Goal: Show that $p \to q, q \to r \vdash p \to r$

$$\begin{array}{c} ? \\ \vdots \\ p \to r \end{array}$$

Example 2: Hypothetical Syllogism

Goal: Show that $p \to q, q \to r \vdash p \to r$

$$\dfrac{\begin{matrix}[p] \\ \vdots \\ r\end{matrix}}{p \to r} \to I$$

Example 2: Hypothetical Syllogism

Goal: Show that $p \to q, q \to r \vdash p \to r$

$$\cfrac{\cfrac{\begin{array}{c}?\\ \vdots \\ q\end{array} \qquad q \to r}{r}\ \to E}{p \to r}\ \to I$$

Example 2: Hypothetical Syllogism

Goal: Show that $p \to q, q \to r \vdash p \to r$

$$\dfrac{\dfrac{\dfrac{[p] \qquad p \to q}{q} \to E \qquad q \to r}{r} \to E}{p \to r} \to I$$

Example 3: Currying

Goal: Show that $p \to (q \to r) \vdash (p \wedge q) \to r$

$$\dfrac{\begin{array}{c}[p \wedge q] \\ \vdots \\ r\end{array}}{(p \wedge q) \to r} \to I$$

Example 3: Currying

Goal: Show that $p \to (q \to r) \vdash (p \wedge q) \to r$

$$\dfrac{\dfrac{\begin{array}{c} ? \\ \vdots \\ q \end{array} \qquad \begin{array}{c} ? \\ \vdots \\ q \to r \end{array}}{r} \to E}{(p \wedge q) \to r} \to I$$

Example 3: Currying

Goal: Show that $p \to (q \to r) \vdash (p \wedge q) \to r$

$$
\dfrac{\dfrac{\dfrac{[p \wedge q]}{q}\ \wedge E_R \qquad \dfrac{\begin{array}{c}?\\ \vdots\\ p\end{array} \qquad p \to (q \to r)}{q \to r}\ \to E}{r}\ \to E}{(p \wedge q) \to r}\ \to I
$$

Example 3: Currying

Goal: Show that $p \rightarrow (q \rightarrow r) \vdash (p \wedge q) \rightarrow r$

$$\dfrac{\dfrac{\dfrac{[p \wedge q]}{q}\ \wedge E_R \qquad \dfrac{\dfrac{[p \wedge q]}{p}\ \wedge E_L \qquad p \rightarrow (q \rightarrow r)}{q \rightarrow r}\ \rightarrow E}{r}\ \rightarrow E}{(p \wedge q) \rightarrow r}\ \rightarrow I$$

Example 4: Contrapositive

Goal: Show that $p \to q \vdash \neg q \to \neg p$

$$\dfrac{\begin{matrix}[\neg q] \\ \vdots \\ \neg p\end{matrix}}{\neg q \to \neg p} \to I$$

Example 4: Contrapositive

Goal: Show that $p \to q \vdash \neg q \to \neg p$

$$\dfrac{\dfrac{\begin{matrix}[p] \\ \vdots \\ \psi\end{matrix} \qquad \begin{matrix}[p] \\ \vdots \\ \neg\psi\end{matrix}}{\neg p}\ \neg I}{\neg q \to \neg p}\ \to I$$

Example 4: Contrapositive

Goal: Show that $p \to q \vdash \neg q \to \neg p$

$$
\dfrac{\dfrac{\dfrac{[p] \qquad p \to q}{q}\ \to E \qquad [\neg q]}{\neg p}\ \neg I}{\neg q \to \neg p}\ \to I
$$

Example 5: De Morgan's Law

Goal: Show that $\neg(p \vee q) \vdash \neg p \wedge \neg q$

$$\dfrac{\begin{array}{c} ? \\ \vdots \\ \neg p \end{array} \qquad \begin{array}{c} ? \\ \vdots \\ \neg q \end{array}}{\neg p \wedge \neg q} \; \wedge I$$

Example 5: De Morgan's Law

Goal: Show that $\neg(p \vee q) \vdash \neg p \wedge \neg q$

$$
\dfrac{
\dfrac{\begin{matrix}[p] & [p]\\ \vdots & \vdots \\ \psi & \neg\psi\end{matrix}}{\neg p}\,\neg I
\qquad
\dfrac{\begin{matrix}[q] & [q]\\ \vdots & \vdots\\ \chi & \neg\chi\end{matrix}}{\neg q}\,\neg I
}{\neg p \wedge \neg q}\,\wedge I
$$

Example 5: De Morgan's Law

Goal: Show that $\neg(p \vee q) \vdash \neg p \wedge \neg q$

$$\dfrac{\dfrac{\dfrac{[p]}{p \vee q}\ \vee I_L \qquad \neg(p \vee q)}{\neg p}\ \neg I \qquad \dfrac{\dfrac{[q]}{p \vee q}\ \vee I_R \qquad \neg(p \vee q)}{\neg q}\ \neg I}{\neg p \wedge \neg q}\ \wedge I$$

Example 6: Law of Excluded Middle

Goal: Show that $\vdash p \vee \neg p$

$$
\dfrac{\begin{array}{c}[\neg(p \vee \neg p)] \\ \vdots \\ \psi\end{array} \qquad \begin{array}{c}[\neg(p \vee \neg p)] \\ \vdots \\ \neg\psi\end{array}}{p \vee \neg p} \ \neg E \text{ (RAA)}
$$

Example 6: Law of Excluded Middle

Goal: Show that $\vdash p \vee \neg p$

$$\dfrac{\begin{array}{c} ? \\ \vdots \\ p \vee \neg p \end{array} \qquad [\neg(p \vee \neg p)]}{p \vee \neg p} \; \neg E \text{ (RAA)}$$

Example 6: Law of Excluded Middle

Goal: Show that $\vdash p \vee \neg p$

$$\dfrac{\dfrac{\begin{array}{c} ? \\ \vdots \\ p \end{array}}{p \vee \neg p}\ \vee I_L \qquad [\neg(p \vee \neg p)]}{p \vee \neg p}\ \neg E \text{ (RAA)}$$

Example 6: Law of Excluded Middle

Goal: Show that $\vdash p \vee \neg p$

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{[p]}{p \vee \neg p}\,\vee I_L \qquad [\neg(p \vee \neg p)]
    }{\neg p}\,\neg I
  }{p \vee \neg p}\,\vee I_R
  \qquad [\neg(p \vee \neg p)]
}{p \vee \neg p}\,\neg E \text{ (RAA)}
$$

Exercises

Try to construct Natural Deduction proofs for the following valid inferences:

1. $p \to (q \to r), p \to q \vdash p \to r$
2. $(p \to q) \to p \vdash p$ (Peirce's Law, requires RAA)
3. $\neg p \vee \neg q \vdash \neg(p \wedge q)$ (The other De Morgan's Law)
4. $p \vee q, \neg p \vdash q$ (Disjunctive Syllogism)