Mathematical Logic

Lecture 3: Propositional Logic –Syntax & Semantics

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Formal Language

We call expressions like $p \vee q \vee r$ and $(\neg K \vee S) \rightarrow A$ formulas of a formal language.

With the help of predefined symbols and parentheses, a formal language avoids the ambiguities present in natural language.

Formal Language

Definition (Propositional Logic Formula)

- Every propositional symbol $(p, q, r, \ldots)$ is a formula .
- If φ is a formula, then $(\neg\varphi)$ is a formula .
- If φ_1 and φ_2 are formulas, then $(\varphi_1 \vee \varphi_2)$, $(\varphi_1 \wedge \varphi_2)$, $(\varphi_1 \rightarrow \varphi_2)$, and $(\varphi_1 \leftrightarrow \varphi_2)$ are formulas .
- Nothing else is a formula.

Formal Language

Regarding the definition of a Propositional Logic Formula :

- What if we replaced "φ", "φ_1", and "φ_2" in the definition with "p", "p_1", and "p_2"? Would that work?
- Meta-language vs. Object language

Formal Language

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Regarding the definition of a Propositional Logic Formula :

- Why must we add the clause "Nothing else is a formula"?
- This type of definition is known as an inductive definition or recursive definition .

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Formal Language

Example (Inductive Definition of Addition)

Suppose a machine only knows how to add 1 ("+1"). How can we "teach" it general addition $(n + m = ?)$?

- Base case: $n + 0 = n$
- Inductive step: $n + (m + 1) = (n + m) + 1$

Formal Language

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Formal Language

Parenthesis Omission Rules

- The outermost parentheses of the entire formula can be omitted.
- The negation symbol $\neg$ has the highest priority (the strongest binding power).
- Consecutive identical logical symbols are conventionally grouped from the right (right-associative).

Formal Language

Example (Parenthesis Omission)

- $(\neg p \vee q) \rightarrow r$
- $p \vee (q \vee (r \vee s))$ is written as $p \vee q \vee r \vee s$
- $p \rightarrow (q \rightarrow (r \rightarrow s))$ is written as $p \rightarrow q \rightarrow r \rightarrow s$

Example (The construction tree of a formula)

$$((\neg p) \vee q) \to r$$

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$$((\neg p) \vee q) \to r$$

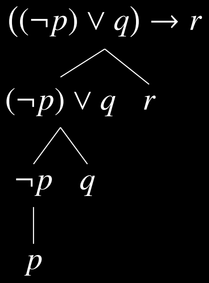
$$(\neg p) \vee q \qquad r$$

Example (The construction tree of a formula)

$$
\begin{tikzcd}[column sep=small]
 & & ((\neg p) \vee q) \to r \arrow[dl, no head] \arrow[dr, no head] & \\
 & (\neg p) \vee q \arrow[dl, no head] \arrow[dr, no head] & & r \\
\neg p & & q &
\end{tikzcd}
$$

Formal Language

Example (The construction tree of a formula)



Formal Language

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$$p \vee q$$

Formal Language

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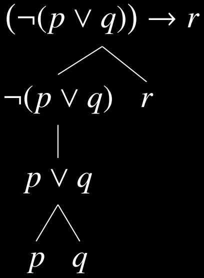

Natural vs. Formal Language

- Early Analytic Philosophers & Logicians: Natural language is systematically misleading. Strict science should (in principle) use formal language.
- Ordinary Language Philosophy
- Reality:
  - Mathematicians (and logicians) still use natural language and abundant shorthand to write proofs.
  - Computer scientists develop programming languages that increasingly resemble natural language.

Syntax of a Language

So far, we have discussed:

- The use of symbols (which act as variables for propositions, and which are constants for logical connectives).
- The construction of formulas and their rules.

These belong to the syntax of the language. They have nothing to do with the specific meaning of the language.

For example, in terms of pure syntax, there is no structural difference between $\vee, \wedge, \rightarrow, \leftrightarrow$.

Semantics of Propositional Logic

Recall: In a specific proposition or inference, we choose our atomic propositions based on our needs. Once the number of atomic propositions is set, we can easily calculate the number of possible truth-value combinations. We call each combination a semantic situation .

Semantics of Propositional Logic

Example

$(\neg p \vee q) \to r$ contains three propositional variables, representing three mutually independent basic propositions (also called atomic formulas). Their semantic situations are:

$$\{pqr, pq\bar{r}, p\bar{q}r, p\bar{q}\bar{r}, \bar{p}qr, \bar{p}q\bar{r}, \bar{p}\bar{q}r, \bar{p}\bar{q}\bar{r}\}$$

Semantics of Propositional Logic

We can view a semantic situation as a **valuation function**.

For instance, the situation $p\bar{q}r$ can be seen as a function $V_{p\bar{q}r}$:

- $V_{p\bar{q}r}(p) = T$
- $V_{p\bar{q}r}(q) = F$
- $V_{p\bar{q}r}(r) = T$

Semantics of Propositional Logic

We can view a semantic situation as a valuation function.

For instance, the situation $p\bar{q}r$ can be seen as a function $V_{p\bar{q}r}$:

- $V_{p\bar{q}r}(p) = 1$
- $V_{p\bar{q}r}(q) = 0$
- $V_{p\bar{q}r}(r) = 1$

Semantics of Propositional Logic

The basic setting of Classical Propositional Semantics:

- In any given situation, a proposition is either true or false. (Bivalence)

  This applies whether it is an atomic proposition or a complex proposition / formula.
- The semantic value of a proposition is exclusively its truth value.

Semantics of Propositional Logic

How do we express the semantics of a logical constant (i.e., a logical connective)?

- The semantics of a connective are the rules for determining the truth value of a complex proposition based on the truth values of its atomic components.

Semantics of Propositional Logic

Truth Table

φ	$\neg\varphi$
1	**0**
0	**1**

Semantics of Propositional Logic

Truth Table

φ	ψ	$\varphi \wedge \psi$	$\varphi \vee \psi$	$\varphi \rightarrow \psi$	$\varphi \leftrightarrow \psi$
1	1	**1**	**1**	**1**	**1**
1	0	**0**	**1**	**0**	**0**
0	1	**0**	**1**	**1**	**0**
0	0	**0**	**0**	**1**	**1**

Material Implication

Note: According to the truth table for $\varphi \to \psi$, as long as φ is false or ψ is true, $\varphi \to \psi$ evaluates to true. This semantic interpretation of "$\to$" is called material implication .

Example

- Sailing west can reach India $\to$ The Earth is round.
- The Nazis won WWII $\to$ Everyone is happy.

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Material Implication

In the semantics of propositional logic, we only care about truth values.

φ	ψ	$\varphi \rightarrow \psi$
1	1	**1**
1	0	**0**
0	1	?
0	0	?

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Material Implication

Example

- For any given number, if it is greater than 3, then it is greater than 2.
- If I become the CEO of OpenAI, I will open-source GPT-5 tomorrow.
- If the Chinese football team reaches the World Cup top 16, I will give everyone an A.

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Material Implication

Example (The "Goldbach Conjecture" AI Trick)

This is a number theory question. If 9301910461290695703809730523177 is not prime, then the Goldbach conjecture holds. Is this true?

- LLMs often fail this logic test because they try to find a mathematical causal link, ignoring the strict propositional logic definition of material implication.

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Material Implication & LLMs

ChatGPT's Response (Link)

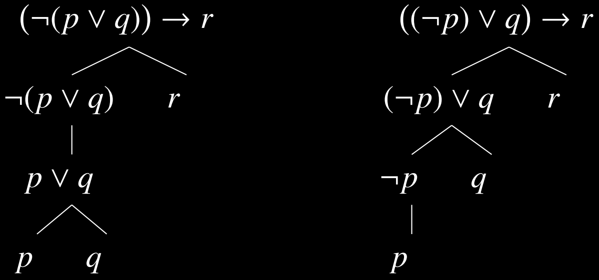
No. As stated, the implication If 9301910461290695703809730523177 is not prime, then the Goldbach conjecture holds. is not true (or at least there is no known mathematical reason for it to be true).

Here's why: The primality or compositeness of a single specific integer is a completely local fact. The Goldbach conjecture is a universal statement about every even integer greater than 2. There is no known theorem linking the compositeness of that particular 31-digit number to the truth of Goldbach.

So the implication (number is composite) $\Rightarrow$ Goldbach does not hold in ordinary mathematics. Direct answer: No.

Evaluating with Truth Tables

The truth table tells us how to recursively calculate the truth value of complex formulas given the truth values of the atomic propositions.



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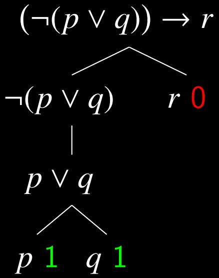

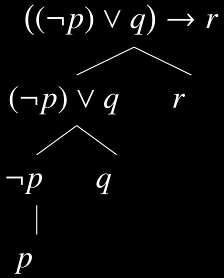

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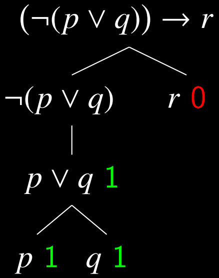

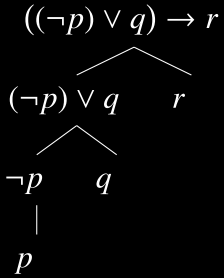

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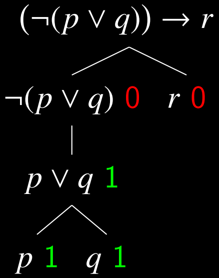

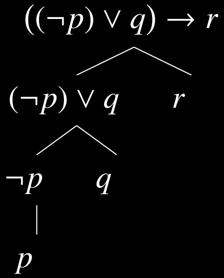

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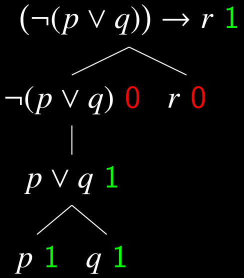

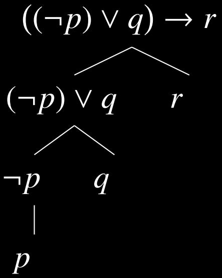

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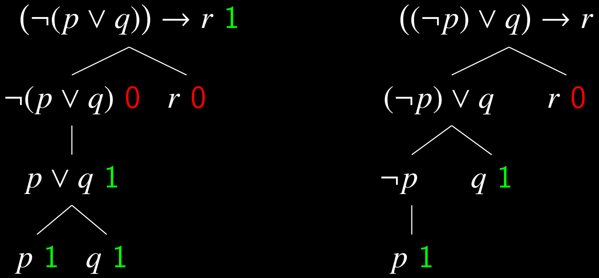

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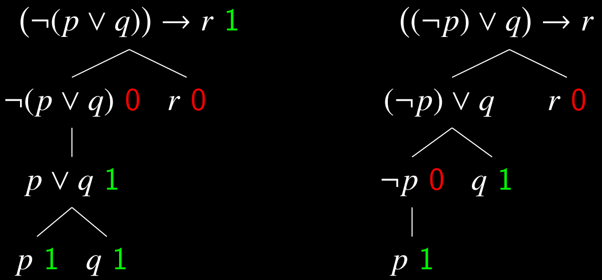

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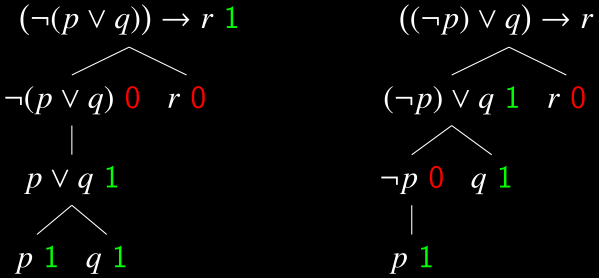

Evaluating with Truth Tables

The truth table tells us how to recursively calculate the truth value of complex formulas given the truth values of the atomic propositions.

$(\neg(p \vee q)) \rightarrow r$ 1

$\neg(p \vee q)$ 0 $\quad r$ 0

$p \vee q$ 1

p 1 $\quad q$ 1

$((\neg p) \vee q) \rightarrow r$ 0

$(\neg p) \vee q$ 1 $\quad r$ 0

$\neg p$ 0 $\quad q$ 1

p 1

Evaluating with Truth Tables

p	q	r	$(\neg(p \vee q)) \to r$	$((\neg p) \vee q) \to r$
1	1	1	1	1
1	1	0	**1**	**0**
1	0	1	1	1
1	0	0	1	1
0	1	1	1	1
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Evaluating with Truth Tables

Recall: A semantic situation in propositional logic is a valuation function V. For example:

$$V(p) = 1,\; V(q) = 1,\; V(r) = 0$$

According to the truth tables for connectives, we can uniquely extend V to evaluate any formula containing only the symbols p, q, r. For example:

$$V((\neg(p \vee q)) \to r) = 1 \qquad V(((\neg p) \vee q) \to r) = 0$$

Evaluating with Truth Tables

Recall: A semantic situation in propositional logic is a valuation function V^{110}. For example:

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According to the truth tables for connectives, we can uniquely extend V to evaluate any formula containing only the symbols p, q, r. For example:

$$V^{110}((\neg(p \vee q)) \to r) = 1 \qquad V^{110}(((\neg p) \vee q) \to r) = 0$$

Validity in Propositional Logic

Definition

Given propositional formulas $\varphi_1, \ldots, \varphi_n$ and ψ, we say the inference from $\varphi_1, \ldots, \varphi_n$ to ψ is valid , or ψ is a (valid) consequence of $\varphi_1, \ldots, \varphi_n$, if and only if , if $V(\varphi_1) = \cdots = V(\varphi_n) = 1$, then $V(\psi) = 1$.

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$$\varphi_1, \ldots, \varphi_n \vDash \psi$$

Validity in Propositional Logic

We can use a truth table to check if an inference is valid.

Example

$$\varphi \to \psi, \neg\psi \vDash \neg\varphi$$

φ	ψ	$\varphi \to \psi$	$\neg\psi$	$\neg\varphi$
1	1	**1**	**0**	**0**
1	0	**0**	**1**	**0**
0	1	**1**	**0**	**1**
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Validity in Propositional Logic

There is a dual concept to validity: Satisfiability .

Definition

A set of propositional formulas $\{\varphi_1, \ldots, \varphi_n\}$ is said to be satisfiable if and only if there exists a valuation function (semantic situation) V such that $V(\varphi_i) = 1$ for all $i = 1, \ldots, n$.

Validity in Propositional Logic

It is not difficult to see that:

$$\varphi_1, \ldots, \varphi_n \vDash \psi$$

if and only if the set:

$$\{\varphi_1, \ldots, \varphi_n, \neg\psi\}$$

is not satisfiable.

Tautologies

Definition

We use $\vDash \varphi$ to denote that φ is a valid consequence of the empty set $\emptyset \vDash \varphi$, which means $\{\neg\varphi\}$ is unsatisfiable. In this case, we call φ a tautology .

Tautologies

Example (Common Tautologies)

- $\neg(\varphi \wedge \neg\varphi)$ Law of Non-Contradiction
- $\varphi \vee \neg\varphi$ Law of Excluded Middle
- $\neg\neg\varphi \leftrightarrow \varphi$ Double Negation

Tautologies

Example (Common Tautologies)

- $\neg(\varphi \lor \psi) \leftrightarrow (\neg\varphi \land \neg\psi)$ De Morgan's Laws

  $\neg(\varphi \land \psi) \leftrightarrow (\neg\varphi \lor \neg\psi)$

- $(\varphi \land (\psi \lor \chi)) \leftrightarrow ((\varphi \land \psi) \lor (\varphi \land \chi))$

  $(\varphi \lor (\psi \land \chi)) \leftrightarrow ((\varphi \lor \psi) \land (\varphi \lor \chi))$ Distributive Laws

Tautologies

Based on the definition, it's easy to see that all valid inferences can be "encoded" as tautologies.

Fact

For any formulas $\varphi_1, \ldots, \varphi_n$ and ψ:

$$\varphi_1, \ldots, \varphi_n \vDash \psi \iff (\varphi_1 \wedge \cdots \wedge \varphi_n) \to \psi \text{ is a tautology}$$

The Limits of Truth Tables

- Truth tables give us a mechanical, decidable algorithm to check validity or satisfiability.
- The Problem: For a formula with n variables, the truth table has 2^n rows.
- If a formula has 100 variables, the truth table has $2^{100} \approx 1.26 \times 10^{30}$ rows. No computer on Earth can compute this by brute force.
- Boolean Satisfiability Problem (SAT): Determining if a formula is satisfiable is the canonical NP-complete problem.

Exercises & Discussion

Translate the following natural language statements into propositional logic formulas:

1. I will only go to school if I get a cookie now.
2. Jack Ma and I are both teachers.
3. Jack Ma and I are billionaires on average.
4. A foreign national is entitled to social security if he has legal employment or if he has had such less than three years ago, unless he is currently also employed abroad.

Exercises & Discussion

Consider the following statement:

Nothing is too trivial to be ignored.

Given a thing x, what can we conclude about x from the above statement? Can you translate it into a propositional logic formula?

Exercises & Discussion

(Considering the parenthesis omission rules), which of the following are valid propositional logic formulas and which are not?

1. $p \to \neg q$
2. $\neg\neg \wedge q \vee p$
3. $p \neg q$
4. $(p \wedge q) \to \neg q$

If it is a formula, draw its construction tree. If not, explain why.

Exercises & Discussion

Can you design a computer program to determine whether a string is a valid propositional logic formula?

- You may redefine what constitutes a propositional logic formula, as long as it is "spiritually consistent" with our definition.
- You may choose to include or exclude the parenthesis omission rules.

Exercises & Discussion

Calculate and construct the truth tables for the following formulas:

1. $(p \rightarrow q) \vee (q \rightarrow p)$
2. $((p \vee \neg q) \wedge r) \leftrightarrow (\neg(p \wedge r) \vee q)$

Exercises & Discussion

Ignoring the parenthesis omission rules, use a truth table to analyze how many non-equivalent propositional logic formulas can be formed by inserting parentheses into the following string:

$$\neg p \to q \vee r$$

Exercises & Discussion

Use a truth table to determine if the following inferences are valid:

- $\neg p \to (q \vee r), \neg q \vDash p \wedge r$
- $\neg p \to (q \vee r), \neg q \vDash p \vee r$

Exercises & Discussion

Use a truth table to determine if the following inferences are valid:

- $p \rightarrow (q \wedge r), \neg q \vDash \neg p$
- $p \rightarrow (q \vee r), \neg q \vDash \neg p$

Exercises & Discussion

Construct the truth tables for the following formulas:

- $(p \vee q) \vee \neg(p \vee (q \wedge r))$
- $(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$
- $((p \leftrightarrow q) \wedge (\neg q \rightarrow r)) \leftrightarrow (\neg(p \leftrightarrow r) \rightarrow q)$

Exercises & Discussion

We say two formulas φ and ψ are logically equivalent (denoted as $\varphi \models\mid \psi$), if and only if $\varphi \models \psi$ and $\psi \models \varphi$. Use truth tables to determine if the following pairs of formulas are logically equivalent:

- $\neg(\varphi \to \psi)$ and $\varphi \vee \neg\psi$
- $\neg(\varphi \to \psi)$ and $\varphi \wedge \neg\psi$

Exercises & Discussion*

The truth table of a propositional logic formula with n propositional symbols can be viewed as an n-ary function.

Example

Suppose $\alpha = ((\neg p) \vee q) \to r$. Define a function B_α such that for every valuation V:

$$B_\alpha(V(p), V(q), V(r)) = V(\alpha)$$

Exercises & Discussion*

1. What is the domain of the function B_α?
2. Do there exist different formulas α, β such that B_α and B_β represent the exact same function?
3. Considering all formulas α containing at most n propositional symbols, how many different B_α functions are there?