Mathematical Logic

Lecture 2: Propositional Logic –Syntax & Translation

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Logic in Everyday Life

- Pork Rice or Fried Rice or Noodles. not Pork Rice, not Fried Rice $\Rightarrow$ Noodles.
- Power broken or Cable broken or Phone broken. not Power broken, not Phone broken $\Rightarrow$ Cable broken.
- 1 or 2 or 3. not 1, not 2 $\Rightarrow$ 3.

More Inferences

If you take the medicine, you will get better.

You took the medicine.

You got better.

More Inferences

If you take the medicine, you will get better.

You got better.

You took the medicine.

More Inferences

If you take the medicine, you will get better.

You didn't take the medicine.

You won't get better.

More Inferences

If you take the medicine, you will get better.

You didn't get better.

You didn't take the medicine.

More Inferences

If I skip class, I will fail.

I did not skip class.

I will not fail.

Valid vs. Invalid Inferences

Clearly, some inferences "make sense", while others "do not".

Convention

If an inference always leads to a true conclusion whenever all its premises are true, we call this inference valid.

Valid vs. Invalid Inferences

In other words, a valid inference has no counter-example—a situation where the premises are all true but the conclusion is false.

If such a situation exists, we call the inference invalid .

Valid vs. Invalid Inferences

Wait a minute,

- What exactly is the inference we are talking about here? Is it just a bunch of sentences? Or the facts they represent?
- When we say "valid" or "invalid", what property of the inference are we describing?

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Valid vs. Invalid Inferences

Recall:

If you take the medicine, you will get better.

You got better.

You took the medicine.

Valid vs. Invalid Inferences

Recall:

If you take the medicine, you will get better.	—True
You got better.	—True
You took the medicine.	—True

Valid vs. Invalid Inferences

Recall:

If you take the medicine, you will get better.	—False
You got better.	—True
You took the medicine.	—False

Valid vs. Invalid Inferences

Recall:

If you take the medicine, you will get better.	—True
You got better.	—True
You took the medicine.	—False

Valid vs. Invalid Inferences

- First, we are looking at a set of propositions, divided into several premises and one conclusion. They can be true or false, and there may be certain constraints on their combinations of truth values.

Valid vs. Invalid Inferences

Example

For instance, an inference with two premises and one conclusion can be written as:

$$\frac{P_1, P_2}{C}$$

The 3 propositions involved have at most ? possible combinations of True and False.

Valid vs. Invalid Inferences

Example

For instance, an inference with two premises and one conclusion can be written as:

$$\frac{P_1, P_2}{C}$$

The 3 propositions involved have at most 8 possible combinations of True and False.

Valid vs. Invalid Inferences

Example

But what if we let P_1, P_2, and C be:

P_1: If you take the medicine, you will get better.

P_2: You took the medicine.

C: You got better.

It seems we no longer have all 8 combinations of true and false?

Valid vs. Invalid Inferences

In a valid inference, we consider all possible truth-value combinations of the premises and the conclusion. In every such combination:

- If all premises are true, the conclusion is also true.
- If the conclusion is false, at least one premise is false.

Valid vs. Invalid Inferences

Therefore, a valid inference helps us:

- Tell us what we *must* accept, given the facts we already accept.
- Refutation : If you can prove the conclusion of a valid inference is wrong, then at least one of its premises must be wrong.

Valid vs. Invalid Inferences

Suppose the following inferences are valid, and suppose you know A is true but G is false:

Example

$$\dfrac{\dfrac{A \qquad \dfrac{A \quad B \quad C}{D}}{E} \qquad \dfrac{E}{F} \qquad \dfrac{A}{B}}{G}$$

Do you know whether B, C, D, E, F are true or false?

Valid vs. Invalid Inferences

Suppose the following inferences are valid, and suppose you know A is true but G is false:

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Valid vs. Invalid Inferences

Recall our question: When we say "valid" or "invalid", what property of the inference are we describing?

Recall: A valid inference restricts the possible combinations of true/false propositions. For example:

Look again:

If you take the medicine, you will get better.
You took the medicine.

You got better.

If you didn't take medicine, then you wouldn't get better
You got better

You took the medicine

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Look again:

If I skip class, I will fail
I skipped class

I failed

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You got better

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Recall our question: When we say "valid" or "invalid", what property of the inference are we describing?

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If I skip class, I will fail	If you didn't take medicine, then you wouldn't get better
I skipped class	You got better
I failed	You took the medicine

Valid vs. Invalid Inferences

Recall our question: When we say "valid" or "invalid", what property of the inference are we describing?

Recall: A valid inference restricts the possible combinations of true/false propositions. For example:

Look again:

If I skip class, I will fail
I skipped class

I failed

If the driver hadn't taken a wrong turn,
then WWI wouldn't have started
WWI started

The driver took a wrong turn

Valid vs. Invalid Inferences

Convention

In the previous examples, we say that the two inferences on the left and the two inferences on the right share the same pattern or form. However, the inferences on the left have a different pattern / form from those on the right (though all of them are valid).

Valid vs. Invalid Inferences

- Two inferences with the same form must either both be valid, or both be invalid.
- A major milestone in the origin of logic was the discovery that validity depends primarily on form rather than content.
- Consequently, to preserve the form (and thus preserve validity), some parts of the inference can be freely substituted, while others cannot.

The Use of Symbols

To represent the form of an inference, people use simple symbols to denote the parts that can be substituted. For instance, we use p for "I took the medicine" or "I skipped class", and q for "I got better" or "I failed". We can easily see they share the same form.

If you take the medicine,
then you will get better.
You took the medicine.

You got better.

If p, then q
p

---

q

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If I skip class,
then I will fail
I skipped class

I failed

If p, then q
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If you take the medicine,
then I will fail
You took the medicine.

I failed

If p, then q
p

---

q

The Use of Symbols

Now we can say:

- An inference of the form: from "If p then q" and "p" to "q" is valid.
- An inference of the form: from "If p then q" and "not p" to "not q" is invalid.

The Use of Symbols

Using symbols makes it easier to realize that these forms of inference (even though they involve 3 propositions) only depend on the truth values of 2 "atomic" propositions: p and q. And there are only 4 possible combinations of truth values for p and q.

Thus, for this form of inference, validity / invalidity can be precisely defined as:

> Out of all 4 truth-value combinations for p and q, in every case where the premises are true, the conclusion is also true.

Formalization of Inference

- Formalization helps us determine the validity of an inference.
- To formalize a real-world example, we must define a translation from natural language to formal language.

Formalization of Inference

Example (Formalizing the Waiter)

- Translation from Natural to Formal Language:
  - p : B ordered the pork rice
  - q : B ordered the fried rice
  - r : B ordered the noodles
- Formal Language Version:

  $(p \vee q \vee r, \neg p, \neg q \vDash r)$

Formalization of Inference

Example (Formalizing the Broken Charger)

- Translation from Natural to Formal Language:
  - p : The power supply is broken
  - q : The phone is broken
  - r : The USB cable is broken
- Formal Language Version:

$$(p \vee q \vee r, \neg p, \neg q \vDash r)$$

Formalization of Inference

Example (Formalizing the 3×3 Sudoku)

- Translation from Natural to Formal Language:
  - p : The top-right cell is 1
  - q : The top-right cell is 2
  - r : The top-right cell is 3
- Formal Language Version:

$$(p \vee q \vee r, \neg p, \neg q \vDash r)$$

Formalization of Inference

Example (A deeper formalization of the 3×3 Sudoku)

- Translation from Natural to Formal Language:
  - For any natural numbers $i, j, k \in \{1, 2, 3\}$, let p_{ijk} mean "The cell at row i, column j contains the number k".
- Formal Language Version:
  - For any $i, j \in \{1, 2, 3\}$, $(p_{ij1} \wedge \neg p_{ij2} \wedge \neg p_{ij3}) \vee (\neg p_{ij1} \wedge p_{ij2} \wedge \neg p_{ij3}) \vee (\neg p_{ij1} \wedge \neg p_{ij2} \wedge p_{ij3}) \in \Lambda$

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- Formal Language Version:
  - For any i_1, i_2, j_1, j_2 and $k \in \{1, 2, 3\}$, whenever $i_1 = i_2$ and $j_1 \neq j_2$, or $i_1 \neq i_2$ and $j_1 = j_2$, we have $p_{i_1 j_1 k} \rightarrow \neg p_{i_2 j_2 k} \in \Lambda$

Formalization of Inference

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- Translation from Natural to Formal Language:
  - For any natural numbers $i, j, k \in \{1, 2, 3\}$, let p_{ijk} mean "The cell at row i, column j contains the number k".
- Formal Language Version:
  - $p_{111} \wedge p_{232} \in \Lambda$
  - $\Lambda \vDash p_{133}$

Logical Connectives in Natural Language

Recall: When discussing the form of an inference, we distinguished between the parts that can be replaced (the atomic propositions) and the parts that remain constant.

Example

- Pork Rice or Fried Rice or Noodles, not Pork Rice, not Fried Rice $\Rightarrow$ It is Noodles
- Power broken or Phone broken or Cable broken, not Power broken, not Phone broken $\Rightarrow$ It is Cable broken
- 1 or 2 or 3, not 1, not 2 $\Rightarrow$ It is 3

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Logical Connectives in Natural Language

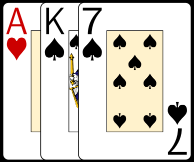
What does the following sentence mean?

She has an Ace if she is missing a King or Spades.

Logical Connectives in Natural Language

Example

In a game of Texas Hold'em, the flop comes with three community cards:



A player evaluates her opponent's strong bet: "She must have Top Pair (Ace), Middle Pair (King), or a Flush Draw (Spades). So, she has an Ace if she is missing a King or Spades."

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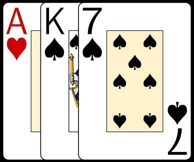

A player evaluates her opponent’s strong bet: ”She must have Top Pair (Ace), Middle Pair (King), or a Flush Draw (Spades). So, she has an Ace if she is missing a King or Spades.”

Logical Connectives in Natural Language

We can borrow parentheses to help clarify the meaning and eliminate ambiguity:

- If she is missing a King or Spades, then she has an Ace.
  - Interpretation 1: If (she is missing a King) or (she is missing Spades), then she has an Ace.
  - Interpretation 2: If she is missing (a King or Spades) [i.e. she has neither], then she has an Ace.

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Logical Connectives in Natural Language

Example

Optional questions on an exam:

You may answer Question 3 or answer Question 4.

Does this "or" mean you can answer both? (Inclusive vs Exclusive OR)

Logical Connectives in Natural Language

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Logical Connectives in Natural Language

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Logical Connectives in Natural Language

Example

You are playing the game *Hearts of Iron* as Finland, facing the Soviet threat:

> You can ally with the Axis or ally with the Allies.

Here, the "or" is strictly exclusive.

Logical Connectives in Natural Language

Example

You are playing the game *Hearts of Iron* as Finland, facing the Soviet threat:

You can ally with the Axis or ally with the Allies.

Here, the "or" is strictly exclusive.

Logical Connectives in Natural Language

One day you find your hard drive encrypted, and receive a message:

"Your data will be lost forever, unless you send 1 Bitcoin to this address."

Does this guarantee that if you send the Bitcoin, your data will be recovered?

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Logical Connectives in Natural Language

Newton's First Law of Motion

> *Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus illud a viribus impressis cogitur statum suum mutare.*
>
> *Philosophiae Naturalis Principia Mathematica*

Logical Connectives in Natural Language

Newton's First Law of Motion

> *Every body perseveres in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed thereon.*

Translation by Andrew Motte

Logical Connectives in Natural Language

Newton's First Law of Motion

> *Every body perseveres in its state of being at rest or of moving uniformly straight forward, except insofar as it is compelled to change its state by the forces impressed.*

Translation by Whitman & Cohen

Logical Connectives in Natural Language

Newton's First Law of Motion

一切物体总保持匀速直线运动状态或静止状态，直到有外力迫使它改变这种状态为止。

Chinese High School Physics Textbook

Logical Connectives

Among the "logical words" in natural language, some are synonyms, some involve different grammatical structures, and some contain unavoidable ambiguities. Therefore, it is necessary to introduce a standardized formal representation.

Logical Connectives

Symbol	Natural Language Equivalent	Term
$\neg$	not, it is not the case that	Negation
$\wedge$	and, but	Conjunction
$\vee$	or	Disjunction
$\rightarrow$	if... then...	Implication
$\leftrightarrow$	if and only if (iff)	Equivalence

Logical Connectives

Recall: We use letters like p, q, r to represent atomic propositions. Thus:

- Do question 3 or question 4
- Ally with the Axis or the Allies

Logical Connectives

Recall: We use letters like p, q, r to represent atomic propositions. Thus:

- $p \vee q$
- Ally with the Axis or the Allies

Logical Connectives

Recall: We use letters like p, q, r to represent atomic propositions. Thus:

- $p \vee q$
- $(p \vee q) \wedge \neg(p \wedge q)$

Logical Connectives

Recall:

- If (not King) or Spades, then Ace
- If not (King or Spades), then Ace

Logical Connectives

Recall:

- $((\neg K) \vee S) \to A$
- If not (King or Spades), then Ace

Logical Connectives

Recall:

- $((\neg K) \vee S) \rightarrow A$
- $(\neg(K \vee S)) \rightarrow A$

Logical Connectives

Recall:

- Newton's First Law:

  If there is no external force, then it remains at rest or in uniform motion

Logical Connectives

Recall:

- Newton's First Law:

  $\neg f \to s$

Logical Connectives

Recall:

- Newton's First Law:

  $\neg s \rightarrow f$

Atomic Propositions

Notice that in the formalization of Newton's First Law, we didn't further decompose "at rest or in uniform motion" into "$r \vee u$".

We said we use letters for "atomic propositions". But what exactly is an atomic proposition?

Logical Atomism

The world can be broken down into ultimate logical facts (or logical atoms) that are indivisible and independent of each other.

An outdated philosophical theory

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Atomic Propositions

Convention: When translating natural language, we determine atomic propositions based on our needs, trying to keep them mutually independent.

Exercises & Discussion

Long-term Project (5 cases per person)

Look for logical phenomena in real life, and try to formalize the logical inferences. Please provide:

- The scenario (context), the problem, and the answer in natural language.
- The translation from natural language to formal language.
- The logical inference written in formal language.

Exercises & Discussion

Long-term Project (5 cases per person)

Grading Criteria

- The inference in both natural and formal languages must be valid (or genuinely invalid if claimed to be).
- The natural language inference must be natural.
- Is your example novel (compared to our existing database)?
- Can your example "fool" mainstream LLMs?

Exercises & Discussion

Long-term Project (5 cases per person)

Submit your work:

`https://logic.fudan.edu.cn/logictranslation`

Exercise: Texas Hold'em Logic

We discussed two interpretations for:

"She has an Ace if she is missing a King or Spades."

- Interpretation 1: $(\neg K \vee \neg S) \to A$
- Interpretation 2: $\neg(K \vee S) \to A$

Can you design a Texas Hold'em scenario (a specific flop and betting context) where Interpretation 1 is the only reasonable logical deduction?

Exercises & Discussion

- Recall the "Waiter serving food" example. For the waiter, what were all the initial possible combinations? What happened when you (C) answered? What happened when A answered?

- Recall the "3×3 Sudoku" example. If we remove the "2" from the right cell, is it still a valid Sudoku puzzle? Why? How many valid completed 3×3 grids exist in total?

1		
		2

Exercises & Discussion

Logic helps us not only solve Sudoku, but also construct Sudoku puzzles.

Given any completed, valid 9×9 grid, we randomly remove the number in one cell. It is now a Sudoku puzzle, just a very easy one. Then we randomly remove another number... When we remove a number and it no longer has a unique solution, we put it back. By doing this, we get a decent Sudoku puzzle. We can try removing other numbers until removing ANY remaining number destroys the uniqueness of the solution. Then we have a challenging Sudoku puzzle.

Can you discuss the logic behind this process?

Appendix: Rules of Classic Sudoku

- Generally composed of nine 3×3 subgrids.
- Each column must contain the numbers $1 \sim 9$, with no omissions and no repetitions.
- Each block (the area enclosed by thick black lines, usually a 3×3 subgrid) must contain the numbers $1 \sim 9$, with no omissions and no repetitions.

Give it a try?

	2			3		9		7
	1							
4		7				2		8
		5	2				9	
			1	8		7		
	4				3			
				6			7	1
	7							
9		3		2		6		5